3 Scalar Field Interactions

We will determine *model Lagrangian, S-matrix, perturbation theory and Diagrammatic representations.*

3.1 Interaction models

There are three types of scalar field interaction:

- self-interaction, with polynomial potential

$$\mathcal{V}(\phi) = \frac{\lambda}{n!}\phi^n(x) \mapsto \phi^n - model \tag{1}$$

  when λ is interaction strength or coupling constant and $n!$ is symmetry factor.

- Yukawa-type interaction, i.e. with potential

$$\mathcal{V}_{Yukawa}(\phi, \Phi) = \lambda\phi\Phi^2 \tag{2}$$

- Coupling with vector field with some *gauge symmetry*, this will be determined later

For our convenient for doing doing some formulation we will use the ϕ^3-self interaction as our sample model.

3.2 S-matrix theory

In quantum field theory we are interested in doing calculation of the transition probability of quantum state of two particle from their interaction, i.e. scattering or collision. Let $|p_1, p_2; \alpha\rangle$ is in the incoming particle state and $|q_1, q_2; \beta\rangle$ be the out going state after interaction. The transition probability is determined from the defined S-matrix which is written in the form

$$|p_1, p_2; \alpha\rangle = S_{\alpha\beta}|q_1, q_2; \beta\rangle \mapsto S_{\alpha\beta} = \langle q_1, q_2; \beta|p_1, p_2; \alpha\rangle \tag{3}$$

Note that the S-operator is unitary, and its details can be analyzed from the action of field operators as in the following. Let us write from above

$$\begin{aligned}
S_{\alpha\beta} = \langle\beta; q_1, q_2|p_1, p_2;\alpha\rangle &\equiv \langle\beta; q_1, q_2|a^\dagger(p_1)|p_2;\alpha\rangle \\
&= -i\int d^3x_1 e^{-ip_1\cdot x_1}\overleftrightarrow{\partial}_{x_1^0}\langle\beta; q_1, q_2|\phi(x_1)|p_2;\alpha\rangle \\
&= i\int d^3x_1\left\{\left(\partial_{x_1^0}e^{-ip_1\cdot x_1}\right)\langle\beta; q_1, q_2|\phi(x_1)|p_2;\alpha\rangle\right. \\
&\qquad \left. -e^{-ip_1\cdot x_1}\partial_{x_1^0}\langle\beta; q_1, q_2|\phi(x_1)|p_2;\alpha\rangle\right\} \\
&\equiv i\int d^4x_1\partial_{x_1^0}\left\{\left(\partial_{x_1^0}e^{-ip_1\cdot x_1}\right)\langle\beta; q_1, q_2|\phi(x_1)|p_2;\alpha\rangle\right. \\
&\qquad \left. -e^{-ip_1\cdot x_1}\partial_{x_1^0}\langle\beta; q_1, q_2|\phi(x_1)|p_2;\alpha\rangle\right\} \\
&= i\int d^4x_1\left\{\left(\partial^2_{x_1^0}e^{-ip_1\cdot x_1}\right)\langle\beta; q_1, q_2|\phi(x_1)|p_2;\alpha\rangle\right. \\
&\qquad \left. -e^{-ip_1\cdot x_1}\partial^2_{x_1^0}\langle\beta; q_1, q_2|\phi(x_1)|p_2;\alpha\rangle\right\} \qquad (4)
\end{aligned}$$

Using the fact that

$$\begin{aligned}
0 = (\partial^2_{x_1} + m_1^2)e^{-ip_1\cdot x_1} &= (\partial^2_{x_1^0} - \nabla^2_{x_1} + m_1^2)e^{-ip_1\cdot x_1} \\
\mapsto \partial^2_{x_1^0}e^{-ip_1\cdot x_1} &= (\nabla^2_{x_1} - m_1^2)e^{-ip_1\cdot x_1}
\end{aligned}$$

The doing integration by part two times of the term containing $\nabla^2_{x_1}$ and ignore all boundaries, we will get from above

$$\begin{aligned}
S_{\alpha\beta} &= -i\int d^4x_1 e^{-ip_1\cdot x_1}(\partial^2_{x_1^0} - \nabla^2_{x_1} + m_1^2)\langle\beta; q_1, q_2|\phi(x_1)|p_2;\alpha\rangle \\
&\equiv -i\int d^4x e^{-ip_1\cdot x_1}(\partial^2_{x_1} + m_1^2)\langle\beta; q_1, q_2|\phi(x_1)|p_2;\alpha\rangle \qquad (5)
\end{aligned}$$

A similar analysis for p_2 state, we will get

$$\begin{aligned}
S_{\alpha\beta} &= (-i)^2\int d^4x_1\int d^2x_2 e^{-ip_1\cdot x_1 - op_2\cdot x_2} \\
&\times(\partial^2_{x_1} + m_1^2)(\partial^2_{x_2} + m_2^2)\langle\beta; q_1, q_2|T[\phi(x_1)\phi(x_2)]|\Omega;\alpha\rangle \qquad (6)
\end{aligned}$$

when the extra time ordering operator T is inserted to take care all possible time ordering inside the two field operators., and we have defined the interacting ground state $|\Omega\rangle$.

A similar process of analysis can be done with state $|q_1, q_2;\beta\rangle$ and results in term of the conjugation as

$$\begin{aligned}
S_{\alpha\beta} = (-i)^4\int d^4x_1\int d^4x_2\int d^4y_1\int d^4y_2\, & e^{-ip_1\cdot x_1 - ip_2\cdot x_2 + iq_1\cdot y_1 + iq_2\cdot y_2} \\
&\times(\partial^2_{x_1} + m_1^2)(\partial^2_{x_2} + m_2^2)(\partial^2_{y_1} + m_3^2)(\partial^2_{y_2} + m_4^2) \\
&\times\langle\beta;\Omega|T[\phi(x_1)\phi(x_2)\phi(y_1)\phi(y_2)]|\Omega;\alpha\rangle \qquad (7)
\end{aligned}$$

This is known in the name of *LSZ* reduction formula, according to *Lehmann, Symanzik and Zimmermann.* Its general form appear as

$$\begin{aligned}\langle q_1, q_2, ..., q_n | p_1, p_2, ..., p_m \rangle = (-i)^{n+m} \int d^4x_1 ... \int d^4x_m \int d^4y_1 ... \int d^4y_n \\ \times \exp\left\{-i\sum_{j=1}^{n} p_j \cdot x_j + i\sum_{l=1}^{n} q_l \cdot y_l\right\} (\partial^2_{x_1} + M_1^2) ... (\partial^2_{x_m} + M_m^2) \\ \times(\partial^2_{y_1} + M_1'^2) ... (\partial^2_{y_n} + M_n'^2) \langle \Omega | T[\phi(x_1) ... \phi(x_m)\phi(y_1) ... \phi(y_n)] | \Omega \rangle \end{aligned} \quad (8)$$

3.3 Perturbation theory

3.3.1 Interaction picture

Note that the field operator $\phi(x)$ is born to be *Heisenberg operator*, satisfy Heisenberg's equation

$$i\partial_0 \phi(x) = [\phi(x), H] \quad (9)$$

where H is the field Hamiltonian operator. With interaction, we can write

$$H = H_0 + V, \text{ where } V = \int d^3x \mathcal{V}(\phi) \quad (10)$$

The Heisenberg field operator can be similarity transformed from Schrodinger field operator as

$$\phi(x) = U^\dagger(t, 0)\phi_S(0, \vec{x})U(t, 0), \text{ where } U(t, 0) = Te^{-i\int_0^t H dt'} \quad (11)$$

On the other hand we also have the field operator in *interacting picture* $\phi_I(x)$ which has similarity transformed from the Schrodinger field operator in the form

$$\phi_I(x) = U_0^\dagger(t, 0)\phi_S(0, \vec{x})U_0(t, 0), \text{ where } U_0(t, 0) = Te^{-i\int_0^t H_0 dt'} \quad (12)$$

Then we can write

$$\phi(x) = U^\dagger(t, 0)U_0(t, 0)\phi_I(x)U_0^\dagger(t, 0)U(t, 0) \quad (13)$$

By the invention

$$\begin{aligned} U_I(t, 0) &= U_0^\dagger(t, 0)U(t, 0) \\ &\equiv Te^{-i\int_0^t V_I dt'} = Te^{-i\int_0^t dt' \int d^3x' \mathcal{V}(\phi_I(x'))} \quad (14) \\ \phi(x) &= U_I^\dagger(t, 0)\phi_I(x)U_I(t, 0) \quad (15) \end{aligned}$$

Now let us determine the product of time-ordered Heisenberg field operators

$$T[\phi(x_1)\phi(x_2)...\phi(x_n)]$$

Assume $t_1 > t_2 > ... > t_n$, for simplicity, one can write it in term of product of interaction field operators as

$$U^\dagger(t_1)\phi_i(x_1)U_I(t_1)U_I^\dagger(t_2)\phi_I(x_2)U_I(t_2)...U_I^\dagger(t_n)\phi_I(x_n)U_I(t_n) \tag{16}$$

With the fact that $U(t) = Te^{-i\int^t dt' \int d^3x' \mathcal{V}(\phi_I(x'))}$ we will have

$$\mapsto U_I^\dagger(t_1)U_I(t_2) = Te^{-i\int_2^1 dt' \int d^3x' \mathcal{V}(\phi_I(x'))} \equiv U_I(1,2) \tag{17}$$

From above, in generic time-ordering, we can write

$$T[\phi(x_1)\phi(x_2)...\phi(x_n)] = T[U(t_1,t_n)\phi_I(x_1)\phi_I(x_2)...\phi_I(x_n)] \tag{18}$$

3.3.2 Gell-Mann and Low theorem

The theorem state that

"*the interaction can be assumed to be adiabatic developed from no-interaction in the far past and adiabatic die out into no-interaction in the far future*"

With this assumption the interaction ground state $|\Omega\rangle$ will become the non-interacting ground state $|0\rangle$ in the far past or far future. Then the non-interaction ground will develop in time by interaction as

$$e^{-iHT}|0\rangle = e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle + \underbrace{\sum_{n\neq 0} e^{-iE_nT}|n\rangle\langle n|0\rangle}_{\to Ignored\ as\ T\to\infty} \tag{19}$$

Then we have

$$\begin{aligned}|\Omega(t_0)\rangle &= \lim_{T\to\infty}\left(e^{-iE_0(t_0+T)}\langle\Omega|0\rangle\right)^{-1} e^{-iH(t_0+T)}|0\rangle \\ &= \lim_{T\to\infty}\left(e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle\right)^{-1} e^{-iH(t_0-(-T))}|0\rangle \\ &= \lim_{T\to\infty}\left(e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle\right)^{-1} U(t_0,-T)|0\rangle\end{aligned} \tag{20}$$

Similarly we will have

$$\langle\Omega(t_0)| = \lim_{T\to\infty}\langle 0|U(T,t_0)\left(e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle\right)^{-1} \tag{21}$$

The factor $\langle\Omega|0\rangle$ is determined from the normalization condition

$$1 = \langle\Omega|\Omega\rangle \lim_{T\to\infty}\left(|\langle 0|\Omega\rangle|^2 e^{-2iE_0T}\right)^{-1}\langle 0|U(T,t_0)U(t_0,-T)|0\rangle$$

$$\mapsto |\langle 0|\Omega\rangle|^2 = \frac{e^{-2iE_0T}}{\langle 0|U(T,-T)|o\rangle}$$

3.3.3 Perturbation expansion

From the expression of the S-matrix, on can write

$$\begin{aligned} S_{\alpha\beta} = \lim_{T\to\infty}(-i)^4 \int d^4x_1 \int d^4x_2 \int d^4y_1 \int d^4y_2 ... \\ \times e^{-ip_1\cdot x_1 - ip_2\cdot x_2 + iq_1\cdot y_1 + iq_2\cdot y_2} ... \\ \times(\partial^2_{x_1} + m_1^2)(\partial^2_{x_2} + m_2^2)(\partial^2_{y_1} + m_3^2)(\partial^2_{y_2} + m_4^2)... \\ \times \frac{\langle 0|T[U_I(T,-T)\phi_I(x_1)\phi_I(x_2)\phi_I(y_1)\phi_I(y_2)]|0\rangle}{\langle 0|U_I(T,-T)|0\rangle} \\ = (-i)^4 \int d^4x_1 \int d^4x_2 \int d^4y_1 \int d^4y_2 e^{-ip_1\cdot x_1 - ip_2\cdot x_2 + iq_1\cdot y_1 + iq_2\cdot y_2} \\ \times(\partial^2_{x_1} + m_1^2)(\partial^2_{x_2} + m_2^2)(\partial^2_{y_1} + m_3^2)(\partial^2_{y_2} + m_4^2)\frac{\mathcal{N}}{\mathcal{D}} \end{aligned} \quad (22)$$

When

$$\mathcal{D} = \langle 0|U_I(\infty,-\infty)|0\rangle = \langle 0|Te^{-i\int d^4x'\mathcal{V}(\phi_I(x'))}|0\rangle = 1 + \sum_{n=1}^{\infty}\mathcal{D}_n \quad (23)$$

$$\mathcal{D}_n = \frac{(-i)^n}{n!}\int d^4x'_1...d^4x'_n\langle 0|T[\mathcal{V}(\phi_I(x'_1))...\mathcal{V}(\phi_I(x'_n))]|0\rangle \quad (24)$$

$$\mathcal{N} = \langle 0|T[U_I(\infty,-\infty)\phi_I(x_1)\phi_I(x_2)\phi_I(y_1)\phi_I(y_2)]|0\rangle = 1 + \sum_{n=1}^{\infty}\mathcal{N}_n \quad (25)$$

$$\begin{aligned} \mathcal{N}_n = \frac{(-i)^n}{n!}\int d^4x'_1 ... \int d^4x'_n\langle 0|T[\mathcal{V}(\phi_I(x'_1)...\mathcal{V}(\phi_I(x'_n) \\ \times\phi_I(x_1)\phi_I(x_2)\phi_I(y_1)\phi_I(y_2)]|0\rangle \end{aligned} \quad (26)$$

In case of ϕ^3-interaction, we will have

$$\mathcal{D}_1 = -i\lambda\int d^4x'\langle 0|T[\phi_I^3(x')]|0\rangle = 0 \quad (27)$$

$$\mathcal{D}_2 = \frac{(-i\lambda)^2}{2!(3!)^2}\int d^4x'_1\int d^4x'_2\langle 0|T[\phi_I^3(x'_1)\phi_I^3(x'_2)]|0\rangle \quad (28)$$

$$\mathcal{D}_3 = 0, \text{ and } etc.$$

and

$$\mathcal{N}_1 = -i\lambda\int d^4x'\langle 0|T[\phi_I^3(x')\phi_I(x_1)\phi_I(x_2)\phi_I(y_1)\phi_I(y_2)]|0\rangle = 0 \quad (29)$$

$$\begin{aligned} \mathcal{N}_2 = \frac{(-i\lambda)^2}{2!(3!)^2}\int d^4x'_1\int d^4x'_2\langle 0|T[\phi_I^3(x'_1)\phi_I^3(x'_2)... \\ \times\phi_I(x_1)\phi_I(x_2)\phi_I(y_1)\phi_I(y_2)]|0\rangle \\ \mathcal{N}_3 = 0, \text{ and } etc. \end{aligned} \quad (30)$$

3.3.4 Wick's theorem

The theorem state that

"*time-ordered product of field operators can be written in terms of all possible normal-ordering and contraction pairs*"

For simple example

$$T[\phi(1)\phi(2)\phi(3)\phi(4)] =: \phi(1)\phi(2)\phi(3)\phi(4) : + : \phi(1)\phi(2) : \overbrace{\phi(3)\phi(4)}$$
$$+ : \phi(1)\phi(3) : \overbrace{\phi(2)\phi(4)} + : \phi(1)\phi(4) : \overbrace{\phi(2)\phi(3)}$$

where

$$\langle 0| : \phi(1)\phi(2) : |0\rangle = 0, \;\; \langle 0| \overbrace{\phi(1)\phi(2)} |0\rangle = \Delta(1,2)$$

From above we will have

$$\begin{aligned} \mathcal{D}_2 = \frac{(-i\lambda)^2}{2!(3!)^2} \int d^4x'_1 \int d^4x'_2 \, \{ \Delta(x'_1, x'_2)\Delta(x'_1, x'_2)\Delta x'_1, x'_2) \\ + \Delta(x'_1.x'_1)\Delta(x'_1, x'_2)\Delta(x'_2, x'_2)\} \end{aligned} \tag{31}$$

$$\begin{aligned} \mathcal{N}_2 = \frac{(-i\lambda)^2}{2!(3!)^2} \int d^4x'_1 \int d^4x'_2 \, \{ [\Delta(x'_1, x'_2)\Delta(x'_1, x'_2)\Delta x'_1, x'_2) \\ + \Delta(x'_1.x'_1)\Delta(x'_1, x'_2)\Delta(x'_2, x'_2)] \, [\Delta(x_1, y_1)\Delta(x_2, y_2) \\ + \Delta(x_1, y_2)\Delta(x_2, y_1) + \Delta(x_1, x_2)\Delta(y_1, y_2)] \\ + [\Delta(x_1, x'_1)\Delta(x_2, x'_1)\Delta(x'_1, x'_2)\Delta(x'_2, y_1)\Delta(x'_2, y_2) \\ + \Delta(x_1, x'_1)\Delta(x'_1, y_1)\Delta(x_2, x'_2)\Delta(x'_2, y_2)\Delta(x'_1, x'_2) \\ + \Delta(x_1, x'_1)\Delta(x'_1, y_2)\Delta(x_2, x'_2)\Delta x'_2, y_1)\Delta(x'_1, x'_2)]\} \end{aligned} \tag{32}$$

3.3.5 Diagrammatic representations

Our above expression look so complicate, but we can simplify by using Feynman diagram representation, by first define the field propagator and coupling vertex diagrams (Feynman rules) as

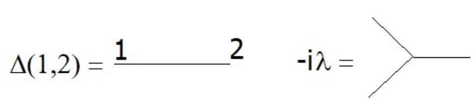

Figure 1: Feynman rules for ϕ^3-interaction

Then we have

Note that denominator terms will appear in from of *disconnected diagrams*, i.e., without legs, while the numerator terms will appear in form of *connected diagrams*. i.e. with legs, factor with disconnected diagrams. This results to the

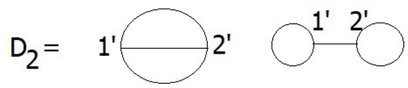

Figure 2: The first non-zero denominator diagrams $\mathcal{D}_2$.

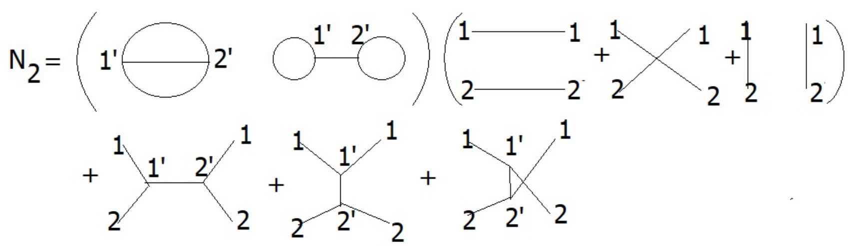

Figure 3: The first non-zero numerator diagrams $\mathcal{N}_2$.

cancellation of the disconnected diagrams, left only with connected diagrams for the S-matrix. By its expression, let us write

$$S_{\alpha\beta} = \sum_{n=0}^{\infty} M_{\alpha\beta}^{(n)} \tag{33}$$

We will have of the zeroth order terms as

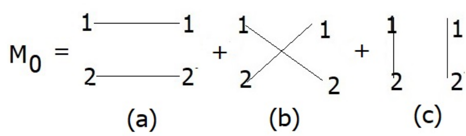

Figure 4: The zero-order term of the S-matrix with connected diagram.

Their expressions are

$$\begin{aligned} M^{(0)} = (-i)^4 \int d^4x_1 \int d^4x_2 \int d^4y_1 \int d^4y_2 e^{-ip_1\cdot x_1 - ip_2\cdot x_2 + iq_1\cdot y_1 + iq_2\cdot y_2} \\ \times(\partial_{x_1}^2 + m^2)(\partial_{x_2}^2 + m^2)(\partial_{y_1}^2 + m^2)(\partial_{y_2}^2 + m^2) \\ \times \{\Delta(x_1, y_1)\Delta(x_2, y_2) + \Delta(x_1, y_2)\Delta(x_2, y_1) + \Delta(x_1, x_2)\Delta(y_1, y_2)\} \quad (34) \\ \equiv M_a^{(0)} + M_b^{(0)} + M_c^{(0)} \quad (35) \end{aligned}$$

And diagrams of the first order terms as

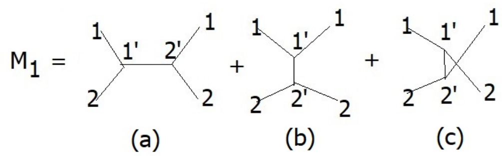

Figure 5: The first-order term of S-matrix with connected diagram.

Their expressions are

$$
\begin{aligned}
M^{(1)} = (-i)^4 \frac{(-i\lambda)^2}{2!(3!)^2} \int d^4x_1 \int d^4x_2 \int d^4y_1 \int d^4y_2 \int d^4x'_1 \int d^4x'_2 ... \\
\times e^{-ip_1\cdot x_1 - ip_2\cdot x_2 + iq_1\cdot y_1 + iq_2\cdot y_2} ... \\
\times(\partial^2_{x_1}+m^2)(\partial^2_{x_2}+m^2)(\partial^2_{y_1}+m^2)(\partial^2_{y_2}+m^2)... \\
\times\{\Delta(x_1,x'_1)\Delta(x_2,x'_1)\Delta(x'_1,x'_2)\Delta(x'_2,y_1)\Delta(x'_2,y_2)... \\
+\Delta(x_1,x'_1)\Delta(x'_1,y_1)\Delta(x_2,x'_2)\Delta(x'_2,y_2)\Delta(x'_1,x'_2)... \\
+\Delta(x_1,x'_1)\Delta(x'_1,y_2)\Delta(x_2,x'_2)\Delta(x'_2,y_1)\Delta(x'_1,x'_2)\} \qquad (36) \\
\equiv M^{(1)}_a + M^{(1)}_b + M^{(1)}_c \qquad (37)
\end{aligned}
$$

3.4 Diagrams on momentum space

Let us determine

$$
\begin{aligned}
M^{(0)}_a = (-i)^4 \int d^4x_1 \int d^4x_2 \int d^4y_1 \int d^4y_2 e^{-ip_1\cdot x_1 - ip_2\cdot x_2 + iq_1\cdot y_1 + iq_2\cdot y_2} \\
\times(\partial^2_{x_1}+m^2)(\partial^2_{x_2}+m^2)(\partial^2_{y_1}+m^2)(\partial^2_{y_2}+m^2)\Delta(x_1,y_1)\Delta(x_2,y_2) \qquad (38)
\end{aligned}
$$

We first apply integration by particle two time of the first two differential operators, and then using the fact that

$$
(\partial^2_{y_1}+m^2)\Delta(x_1,y_1) = i\delta^{(4)}(x_1-y_1),\ (\partial^2_{y_2}+m^2)\Delta(x_2,y_2) = i\delta^{(4)}(x_2-y_2)
$$

Then we have

$$
\begin{aligned}
M^{(0)} = (-i)^2 \frac{1}{p_1^2-m^2}\frac{1}{p_2^2-m^2} \int d^4x_1 \int d^4x_2 e^{-i(p_1-q_1)\cdot x_1 - i(p_2-q_2)\cdot x_2} \qquad (39) \\
= (-i)^2 \frac{1}{p_1^2-m^2}\frac{1}{p_2^2-m^2} \int d^4x_1 \int d^4x_2 e^{-i(p_1-q_1)\cdot x_1 - i(p_2-q_2)\cdot x_2} \\
= \frac{-i}{p_1^2-m^2}\frac{-i}{p_2^2-m^2}(2\pi)^4\delta^{(4)}(p_1-q_1)(2\pi)^4\delta^{(4)}(p_2-q_2) \qquad (40)
\end{aligned}
$$

We get two propagators on momentum space together with their energy-momentum conservation conditions. Similar analysis can be done with $M^{(0)}_b$ and $M^{(0)}_c$, and receive similar results.

Next let us determine $M_a^{(1)}$, we have

$$M_a^{(1)} = (-i)^4 \frac{(-i\lambda)^2}{2!(3!)^2} \int d^4x_1 \int d^4x_2 \int d^4y_1 \int d^4y_2 e^{-ip_1x_1 - ip_2x_2 + iq_1y_1 + iq_2y_2} \times \int d^4x_1' \int d^4x_2' (\partial_{x_1}^2 + m^2)(\partial_{x_2}^2 + m^2)(\partial_{y_1}^2 + m^2)(\partial_{y_2}^2 + m^2) \times \Delta(x_1, x_1')\Delta(x_2, x_1')\Delta(x_1', x_2')\Delta(x_1', y_1)\Delta(x_2', y_2) \quad (41)$$

Using the fact that

$$(\partial_{x_i}^2 + m^2)\Delta(x_i, x_1') = i\delta^{(4)}(x_i - x_1'),\ i = 1, 2$$

$$(\partial_{y_i}^2 + m^2)\Delta(x_2', y_i) = i\delta^{(4)}(x_2' - y_i),\ i = 1, 2$$

We get

$$\begin{aligned} M_a^{(1)} &= \frac{(-i\lambda)^2}{2!(3!)^2} \int d^4x_1' \int d^4x_2' e^{-i(p_1+p_2)\cdot x_1' + i(q_1+q_2)\cdot x_2'} \Delta(x_1', x_2') \\ &= \frac{(-i\lambda)^2}{2!(3!)^2} \underbrace{\int d^4x_1' e^{-i(p+1+p_2-q_1-q_2)\cdot x_1'}}_{=(2\pi)^4\delta^{(4)}(p_1+p_2-q_1-q_2)} \underbrace{\int d^4y e^{-i(q_1+q_2)\cdot y} \Delta(y)}_{=\Delta(q_1+q_2)} \\ &= \frac{(-i\lambda)^2}{2!(3!)^2} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - q_1 - q_2)\Delta(q_1 + q_2) \end{aligned} \quad (42)$$

Note that the factor $2!(3!)^3$ will be canceled by the symmetric number of this diagram, i.e. each diagram will have *symmetric factor* $S^{(n)}$ which will be determined from case to case.

3.5 Feynman rules

From the previous subsection, we observe that we can state some set of rules to write expression from the diagram without any analysis of diagram on configuration space as we have done. This is called *Feynman rules* on momentum space, for the case of real scalar ϕ^3-interaction there are

- internal line particle propagator is $\Delta(p) = \frac{i}{p^2 - m^2 + i\epsilon}$
- interaction vertex is $-i\lambda/3!$, derived from interaction potential $\mathcal{V}(\phi)$
- insert the condition of energy-momentum conservation

  $$(4\pi)^4\delta^{(4)}(p_1 + p_2 - q_1 - q_2),$$

  determined from external legs
- insert the symmetry factor S_n of the n^{th}-order connected diagram

From example of diagram $M_a^{(1)}, M_b^{(1)}$ and $M_c^{(1)}$ above, we will have

$$M_a^{(1)} = (-i\lambda)^2(2\pi)^4\delta^{(4)}(p_1 + p_2 - q_1 - q_2)\Delta(p_1 + p_2) \tag{43}$$
$$M_b^{(1)} = (-i\lambda)^2(2\pi)^4\delta^{(4)}(p_1 + p_2 - q_1 - q_2)\Delta(p_1 - q_1) \tag{44}$$
$$M_c^{(1)} = (-i\lambda)^2(2\pi)^4\delta^{(4)}(p_1 + p_2 - q_1 - q_2)\Delta(p_1 - q_2) \tag{45}$$

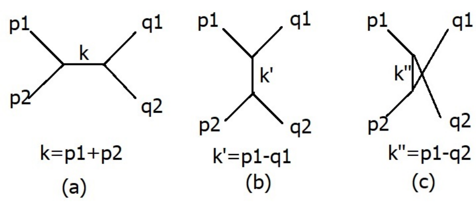

Figure 6: First order connected diagrams on momentum space.

3.6 Mandelstam variables

For convenient to do kinematics evaluation of the interaction process, we always use the Lorentz covariant *Mandelstam variables* which are defined from generic 2-to-2 particle interaction as

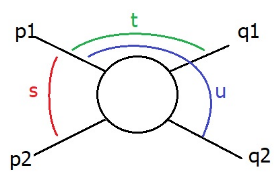

Figure 7: Mandelstam variables.

$$s = (p_1 + p_2)^2 = (q_1 + q_2)^2 \tag{46}$$
$$t = (p_1 - q_1)^2 = (p_2 - q_2)^2 \tag{47}$$
$$u = (p_1 - q_2)^2 = (p_2 - q_1)^2 \tag{48}$$
$$\mapsto s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2 \tag{49}$$

We also deal with Stukelberg's function $\lambda(x, y, z)$ which is defined as

$$\lambda(x, y, z) = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz \tag{50}$$