4 The Cross Section and Decay Rate

In this lecture we come to learn how to compute scattering cross section and decay rate from the S-matrix.

4.1 The cross section

4.1.1 Definition

Let us prepare the incoming state of particles with some momentum distribution of the form

$$|in,\alpha\rangle = \int \frac{d^3p_1}{(2\pi)^3 2E_{p_1}} \int \frac{d^3p_2}{(2\pi)^3 2E_{p_2}} f(p_1)f(p_2)|p_1,p_2;\alpha\rangle \tag{4.1}$$

Then we can rewrite the S-matrix to be in the form

$$\begin{aligned} S_{\alpha\beta} &= \int \frac{d^3p_1}{(2\pi)^3 2E_{p_1}} \int \frac{d^3p_2}{(2\pi)^3 2E_{p_2}} f(p_1)f(p_2)|\langle\beta;q_1,q_2|p_1,p_2;\alpha\rangle & (4.2)\\ &= \delta_{\alpha\beta} + T_{\alpha\beta} & (4.3)\\ \text{with } T_{\alpha\beta} &= \int \frac{d^3p_1}{(2\pi)^3 2E_{p_1}} \int \frac{d^3p_2}{(2\pi)^3 2E_{p_2}} f(p_1)f(p_2)M_{\alpha\beta} & (4.4)\\ M_{\alpha\beta} &= (2\pi)^4\delta^{(4)}(p_1+p_2-q_1-q_2)|\mathcal{M}_{\alpha\beta}|^2 & (4.5) \end{aligned}$$

when $\delta_{\alpha\beta}$ represents amplitude with no interaction amplitude and $T_{\alpha\beta}$ represents amplitude with interaction. The transition probability from interaction will be

$$\begin{aligned} W_{\alpha\beta} = |M_{\alpha\beta}|^2 = &\int \frac{d^3p_1}{(2\pi)^3 2E_{p_1}} \int \frac{d^3p_2}{(2\pi)^3 2E_{p_2}} \int \frac{d^3p'_1}{(2\pi)^3 2E_{p'_1}} \int \frac{d^3p'_2}{(2\pi)^3 2E_{p'_2}} \\ &\times f^*(p'_1)f^*(p'_2)f(p_1)f(p_2)(2\pi)^8\delta^{(4)}(p_1+p_2-q_1-q_2) \\ &\times\delta^{(4)}(p'_1-p'_2-p_1-p_2)|\mathcal{M}_{\alpha\beta}|^2 \end{aligned} \tag{4.6}$$

Using identity

$$(2\pi)^4\delta^{(4)}(p'_1+p'_2-p_1-p_2) = \int d^4x e^{i(p'_1+p'_2-p_1-p_2)\cdot x}$$

$$\int \frac{d^3p'}{(2\pi)^3 2E_{p'}} f^*(p')e^{ip'\cdot x} \equiv \psi^*(x), \text{ assumed with } \omega^2 = E_p^2$$

Assume that the incoming particles have sharp momentum at p_1, p_2, then we can assume (4.6) in the form

$$\begin{aligned} W_{\alpha\beta} &= \int d^4x \frac{|\Psi_1(x)|^2}{2E_1}\frac{|\Psi_2(x)|^2}{2E_2}(2\pi)^4\delta^{(4)}(p_1+p_2-q_1-q_2)|\mathcal{M}_{\alpha\beta}|^2 & (4.7)\\ \mapsto \frac{dW_{\alpha\beta}}{d^3\vec{x}dt} &= \frac{|\Psi_1(x)|^2|\Psi_2(x)|^2}{4E_1E_2}(2\pi)^4\delta^{(4)}(p_1+p_2-q_1-q_2)|\mathcal{M}_{\alpha\beta}|^2 & (4.8)\\ &\equiv d\sigma\times F & (4.9) \end{aligned}$$

This represents *transition density rate* from the interaction, written in terms of differential cross section $d\sigma$ and particle flux density F.

Since $|\Psi(x)|^2 = 2E$, and the flux F is determined from the rest frame of particle 2, $p_1^\mu = (E_1, \vec{p}_1), p_2^\mu = (m_2, \vec{0})$, as

$$\begin{aligned} F = |\Psi(x_1)|^2|\Psi(x_2)|^2 v_{12} = 4E_1 m_2 v_1 = 4E_1 m_2 \frac{|\vec{p}_1|}{E_1} \\ = 4m_2\sqrt{E_1^2 - m_1^2} = 4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} \\ = 2\lambda^{1/2}(s, m_1^2, m_2^2) \end{aligned} \tag{4.10}$$

where s is one of Mandelstam variables and λ is known as Stuckelberg function defined as

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$$

From (4.9), we will get the differential cross section

$$d\sigma = \frac{(2\pi)^2\delta^{(4)}(p_1 + p_2 - q_1 - q_2)|\mathcal{M}|^2}{2\lambda^{1/2}(s, m_1, m_2)} \tag{4.11}$$

4.1.2 Invariant phase space integrals

The Lorentz invariant phase space measure is defined in the form

$$dLIPS = \Pi_{i=1}^n \left[\frac{d^4q_i}{(2\pi)^4}(2\pi)\delta(q_i^2 - m_i'^2)\theta(q_i^0)\right] \tag{4.12}$$

where the non-covariant form is understood as

$$\int \frac{d^4q}{(2\pi)^4}(2\pi)\delta(q^2 - m'^2)\theta(q^0) = \int \frac{d^3q}{(2\pi)^3 2\omega_q}$$

Both forms are alternatively used by convenient.

The total cross section of 2-to-2 particles interaction is then derived from integration overall final state momenta as

$$\begin{aligned} \sigma \quad = \quad & \int \frac{d^3q_1}{(2\pi)^3 2E_3} \int \frac{d^4q_2}{(2\pi)^4}(2\pi)\delta(q_2^2 - m_4^2)\theta(q_2^0) \\ & \times (2\pi)^4\delta^{(4)}\left(p_1 + p_2 - q_1 - q_2\right)\frac{|\mathcal{M}|^2}{2\lambda^{1/2}(s, m_1^2, m_2^2)} \end{aligned} \tag{4.13}$$

4.1.3 Center of mass frame

Let the incoming particles approach each other in 3-direction. So that in the center of mass frame we will have

$$p_1^\mu = (E_1, 0, 0, p) = (\sqrt{p^2 + m_1^2}, 0, 0, p), \tag{4.14}$$

$$p_2^\mu = (E_2, 0, 0, -p) = (\sqrt{p^2 + m_2^2}, 0, 0, -p) \tag{4.15}$$

$$\mapsto s = (E_1 + E_2)^2 = \left(\sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2}\right)^2 \tag{4.16}$$

$$\mapsto p = \frac{\lambda^{1/2}(s, m_1, m_2)}{2\sqrt{s}}, \ E_1 = \frac{s + m_1^2 - m_2^2}{2\sqrt{s}} \tag{4.17}$$

$$E_2 = \frac{s - m_1^2 + m_2^2}{2\sqrt{s}} \tag{4.18}$$

Similarly we will have

$$q_1^\mu = (E_3, \vec{q}) = (\sqrt{q^2 + m_3^2}, q\sin\theta\cos\phi, q\sin\theta\sin\phi, q\cos\theta), \tag{4.19}$$

$$q_2^\mu = (E_4, -\vec{q}) = (\sqrt{q^2 + m_4^2}, -q\sin\theta\cos\phi, -q\sin\theta\sin\phi, -q\cos\theta) \tag{4.20}$$

$$\mapsto s = (E_3 + E_4)^2 = \left(\sqrt{q^2 + m_3^2} + \sqrt{q^2 + m_4^2}\right) \tag{4.21}$$

$$\mapsto q = \frac{\lambda^{1/2}(s, m_3^2, m_4^2)}{2\sqrt{s}}, \ E_3 = \frac{s + m_3^2 - m_4^2}{2\sqrt{s}} \tag{4.22}$$

$$E_4 = \frac{s - m_3^2 + m_4^2}{2\sqrt{s}} \tag{4.23}$$

$$\vec{q} = q\hat{n}, \ \hat{n} = (\sin\theta, \cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{4.24}$$

4.1.4 Differential cross section

Let us determine Mandelstam t-variable

$$t = (p_1 - q_1)^2 = (E_1 - E_3, \vec{p} - \vec{q})^2 = m_1^2 + m_3^2 - 2E_1E_3 + 2|\vec{p}||\vec{q}|\cos\theta \tag{4.25}$$

$$\mapsto dt = 2|\vec{p}||\vec{q}|d\cos\theta \to d\cos\theta = \frac{dt}{2|\vec{p}||\vec{q}|} \tag{4.26}$$

From (4.13) above, let us do d^4q_2 integration using delta function of energy-momentum conservation, we will have

$$\sigma = \frac{1}{(2\pi)^2}\int \frac{d^3q_1}{2E_3}\delta((p_1 + p_2 - q_1)^2 - m_4^2)\frac{|\mathcal{M}|^2}{2\lambda^{1/2}(s, m_1, m_2)} \tag{4.27}$$

Then apply d^3q_1 integration using spherical coordinate

$$d^3q_1 = d\cos\theta d\phi|\vec{q}_1|^2 d|\vec{q}_1|$$

From the argument of the delta function

$$(p_1 + p_2 - q_1)^2 - m_4^2 = (p_1 + p_2)^2 + q_1^2 - 2q_1(p_1 + p_2) - m_4^2$$
$$= s + m_3^2 - 2\sqrt{s}E_3 - m_4^2$$

We also have

$$E_3^2 = |\vec{q}|^2 + m_3^2 \mapsto |\vec{q}|d|\vec{q}| = E_3 dE_3$$

So that we can write the d^3q_1 integral in the form

$$\frac{d^3q_1}{2E_3} = \frac{d\phi dt}{2|\vec{p}||\vec{q}|}|\vec{q}|\frac{E_3}{2E_3}dE_3 = \frac{1}{4|\vec{p}|}d\phi dt dE_3 \tag{4.28}$$

Integrate (4.27), using (4.28) and $\int d\phi = 2\pi$, then we have

$$\frac{d\sigma}{dt} = \frac{1}{8\pi|\vec{p}|}\int dE_3 \delta(s - 2\sqrt{s}E_3 + m_3^2 - m_4^2)\frac{|\mathcal{M}|^2}{\lambda^{1/2}(s, m_1^2, m_2^2)}$$
$$= \frac{|\mathcal{M}|^2}{16\pi|\vec{p}|\sqrt{s}\lambda^{1/2}(s, m_1^2, m_2^2)} \tag{4.29}$$
$$(4.17) \mapsto \frac{d\sigma}{dt} = \frac{1}{16\pi\lambda(s, m_1^2, m_2^2)}|\mathcal{M}|^2 \tag{4.30}$$

From (4.26)

$$\frac{d\sigma}{d\cos\theta} = 2|\vec{p}||\vec{q}|\frac{d\sigma}{dt} \mapsto \frac{d\sigma}{d\Omega} = \frac{1}{2\pi}\frac{d\sigma}{d\cos\theta},\ \sigma = \int\left(\frac{d\sigma}{d\Omega}\right)d\Omega \tag{4.31}$$

4.2 Elastic and inelastic processes

4.2.1 Elastic process

From ϕ^3-interaction at tree level, which is the elastic process, we have

$$\frac{d\sigma}{dt} = \frac{g^4}{16\pi s(s - 4m^2)}\left(\frac{1}{s - m^2} + \frac{1}{t - m^2} + \frac{1}{3m^2 - s - t}\right)$$

After we have used the fact that $s + t + u = 4m^2$, and

$$\mathcal{M}_a = \frac{1}{(p_1 + p_2)^2 - m^2} = \frac{1}{s - m^2}$$

$$\mathcal{M}_b = \frac{1}{(p_1 - q_1)^2 - m^2} = \frac{1}{t - m^2}$$

$$\mathcal{M}_c = \frac{1}{(p_1 - q_2)^2 - m^2} = \frac{1}{u - m^2}$$

4.2.2 Inelastic process

Let us determine the model Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m\phi^2 + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{1}{2}M^2\chi^2 - \frac{g}{2}\chi\phi^2 \tag{4.32}$$

We can set the Feynman rules as

- light scalar field propagator (line): $\Delta_\phi(p) = \frac{i}{p^2-m^2+i\epsilon}$
- heavy scalar field propagator (dash line): $\Delta_\chi(p) = \frac{i}{p^2-M^2+i\epsilon}$
- interaction vertex is $-ig$

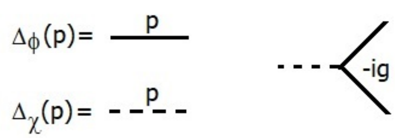

Figure 4.1: Feynman rules of inelastic process.

The tree level diagrams of $\phi\phi \to \chi\chi$ interaction are

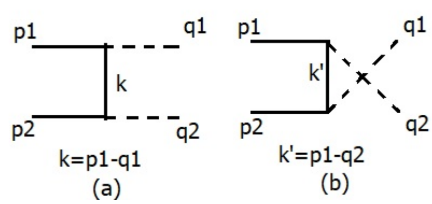

Figure 4.2: Tree diagrams of the $\phi\phi \to \chi\chi$ interaction.

The expressions of the amplitudes are

$$\mathcal{M}_a = \frac{-g^2}{(p_1-q_1)^2-m^2} = \frac{-g^2}{t-m^2}, \tag{4.33}$$

$$\mathcal{M}_b = \frac{-g^2}{(p_1-q_2)^2-m^2} = \frac{-g^2}{u-m^2} = \frac{-g^2}{m^2+2M^2-s-t} \tag{4.34}$$

After we have used the fact that $s+t+u = 2m^2+2M^2$. The cross section is

$$\sigma = \int \frac{d^3q_1}{(2\pi)^3 2E_{q_1}} \int \frac{d^4q_2}{2\pi)^4}(2\pi)(\delta(q_2^2-M^2)\theta(q_2^0) \times (2\pi)^4\delta^{(4)}(p_1+p_2-q_1-q_2)\frac{|\mathcal{M}|^2}{F} \tag{4.35}$$

With

$$s = (p_1 + p_2)^2 = 2m^2 + 2p_1 \cdot p_2$$

$$\to F = 4\sqrt{(p_1 \cdot p_2)^2 - 4m^2} = 2\sqrt{s(s - 4m^2)}$$

We will have from above

$$\sigma = \int \frac{d^3 q_1}{(2\pi)^3 2E_{q1}} \delta\left((p_1 + p_2 - q_1)^2 - M^2\right) \frac{|\mathcal{M}|^2}{2\sqrt{s(2 - 4m^2)}} \tag{4.36}$$

Since

$$|\vec{p}_1| = |\vec{p}_2| = p, |\vec{q}_1| = |\vec{q}_2| = q \mapsto dt = 2pqd\cos\theta$$

$$\delta((p_1 + p_2 - q_1)^2 - M^2) = \delta(s - 2\sqrt{s}E_{q_1})$$

$$p = \frac{1}{2}\sqrt{s - 4m^2}$$

We will end up with

$$\frac{d\sigma}{dt} = \frac{g^4}{16\pi s(s - 4m^2)} \left(\frac{1}{t - m^2} + \frac{1}{m^2 + 2M^2 - s - t}\right)^2 \tag{4.37}$$

4.3 Decay rate

Let us determine the decay of heavy particle $\chi \to \phi\phi$

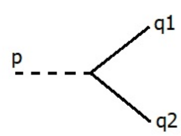

Figure 4.3: The decay $\chi \to \phi\phi$ at first order.

The decay amplitude is

$$M = -ig(2\pi)^4 \delta^{(4)}(p - q_1 - q_2) \mapsto \mathcal{M} = -ig \tag{4.38}$$

The transition probability is determined from Fermi golden rule as

$$W = \int d^4x \frac{|\Psi(x)|^2}{2E_p} |\mathcal{M}|^2 (2\pi)^4 \delta^{(4)}(p - q_1 - q_2) \tag{4.39}$$

The transition rate per unit volume is

$$\frac{dW}{d^3xdt} = d\Gamma \times |\Psi(x)|^2 \tag{4.40}$$

$$\mapsto d\Gamma = \frac{|\mathcal{M}|^2}{2E_p}(2\pi)^4\delta^{(4)}(p - q_1 - q_2) \tag{4.41}$$

$$\text{and } \Gamma = \int \frac{d^3q_1}{(2\pi)^3 2E_{q_1}} \int \frac{d^4q_2}{(2\pi)^4}(2\pi)\delta(q_2^2 - m^2)\theta(q_2^0) \times \frac{|\mathcal{M}|^2}{2E_p}(2\pi)^4\delta^{(4)}(p - q_1 - q_2) \tag{4.42}$$

$$= \frac{g^2}{2E_p}\frac{1}{(2\pi)^2}\int \frac{d^3q_1}{2E_{q_1}}\delta((p - q_1) - m^2) \tag{4.43}$$

Since

$$\frac{d^3q_1}{2E_{q_1}} = \frac{|\vec{q}_1|^2 d|\vec{q}_1| d\Omega}{2E_{q_1}} = \frac{1}{2}|\vec{q}_1| dE_{q_1} d\Omega$$

Integrate $\int d\Omega = 4\pi$, then we have

$$\Gamma = \frac{g^2}{8\pi M}\int dE_{q_1}|\vec{q}_1|\delta(M^2 - 2ME_{q_1}) = \frac{g^2}{16\pi M^2}\sqrt{M^2 - 4m^2} \tag{4.44}$$

This shows that $E_{q_1} = \frac{1}{2}M$ and $|\vec{q}_1| = \frac{1}{2}\sqrt{M^2 - 4m^2}$.