6 Yukawa Interaction

6.1 Interaction Lagrangian

The Yukawa interaction was modeled for nucleon interaction by meson exchange. Nucleon is fermionic field and meson is bosonic field. For p-p interaction neutral meson π^0 exchange, its model Lagrangian will be

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m_p)\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_\pi^2\phi^2 - g\phi\bar{\psi}\psi \tag{6.1}$$

From this Lagrangian we can set the Feynman rules as in the following

- proton propagator (solid line) : $\Delta_p(p) = \frac{i(p\!\!\!/+m_p)}{p^2-m_p^2+i\epsilon}$
- meson propagator (dashed line): $\Delta_\pi(k) = \frac{i}{k^2-m_\pi^2+i\epsilon}$
- interaction vertex: (incoming and outgoing solid lines and dashed line): $-ig$
- incoming proton (incoming solid line): $U(k,s)$
- outgoing proton (outgoing solid line): $\bar{U}(k,s)$
- symmetry factor at each vertex is 1

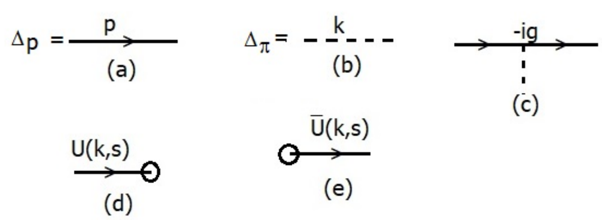

Figure 6.1: Feynman rules for Yukawa interaction.

6.2 Yukawa tree amplitudes

Let us determine $pp \to pp$ interaction amplitude Their expressions are

$$\mathcal{M}_a = (-ig)^2\bar{U}(p_1', s_1')U(p_1, s_1)\frac{i}{k^2 - m_\pi^2 + i\epsilon}\bar{U}(p_2', s_2')U(p_2, s_2) \tag{6.2}$$

$$\mathcal{M}_b = (-ig)^2\bar{U}(p_2', s_2')U(p_1, s_2)\frac{i}{k'^2 - m_\pi^2 + i\epsilon}\bar{U}(p_1', s_1')U(p_2, s_2) \tag{6.3}$$

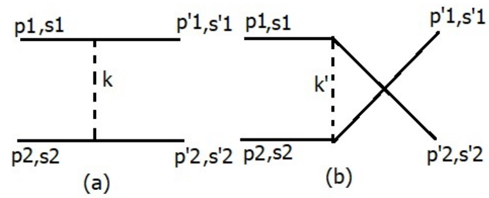

Figure 6.2: Yukawa interaction of $pp \to pp$.

The amplitude squared will be determined with sum overall outgoing spins and averaged overall incoming spins as

$$|\overline{\mathcal{M}}|^2 = \frac{1}{4}\sum_{s1,s2}\sum_{s_1',s_2'}|\mathcal{M}_s + \mathcal{M}_b|^2$$
$$= \frac{1}{4}\sum_{s_1,s_2}\sum_{s_1',s_2'}\left\{\overline{\mathcal{M}}_a\mathcal{M}_a + 2\overline{\mathcal{M}}_a\mathcal{M}_b + \overline{\mathcal{M}}_b\mathcal{M}_b\right\} \tag{6.4}$$

with $k- = p_1 - p_1' \mapsto k^2 = t$, and $k' = p_1 - p_2' \mapsto k'^2 = u$, we have

$$\overline{\mathcal{M}}_a\mathcal{M}_a = \frac{g^4}{(t-m_\pi^2)^2}\left[\bar{U}(p_2,s_2)U(p_2',s_2')\bar{U}(p_2',s_2')U(p_2,s_2)\right]$$
$$\times\left[U(p_1,s_1)\bar{U}(p_1',s_1')\bar{U}(p_1',s_1')U(p_1,s_1)\right] \tag{6.5}$$

$$\mapsto \sum_{s_1,s_2}\sum_{s_1',s_2'}\overline{\mathcal{M}}_a\mathcal{M}_a = \frac{g^4}{(t-m_\pi^2)^2}\sum_{s_2,s_2'}\left[\bar{U}(p_2,s_2)U(p_2',s_2')\bar{U}(p_2',s_2')U(p_2,s_2)\right]$$
$$\times\sum_{s_1,s_1'}\left[U(p_1,s_1)\bar{U}(p_1',s_1')\bar{U}(p_1',s_1')U(p_1,s_1)\right]$$
$$= \frac{g^4}{(t-m_\pi^2)^2}Tr[(\not{p}_2' + m_p)(\not{p}_2 + m_p)]Tr[(\not{p}_1' + m_p)(\not{p}_1 + m_p)] \tag{6.6}$$

Similarly we will have

$$\sum_{s_1,s_2}\sum_{s_1',s_2'}\overline{\mathcal{M}}_b\mathcal{M}_b = \frac{g^4}{(u-m_\pi^2)^2}\sum_{s_1,s_2'}[\bar{U}(p_2,s_2)U(p_1',s_1')\bar{U}(p_1',s_1')U(p_2,s_2)]$$
$$\times[\bar{U}(p_1,s_1)U(p_2',s_2')\bar{U}(p_2',s_2')U(p_1,s_1)]$$
$$= \frac{g^4}{(u-m_\pi^2)^2}Tr[(\not{p}_1' + m_p)(\not{p}_2 + m_p)]Tr[(\not{p}_2' + m_p)(\not{p}_1 + m_p)] \tag{6.7}$$

and

$$\overline{\mathcal{M}}_a\mathcal{M}_b = \frac{g^4}{(t-m_\pi^2)(u-m_\pi^2)}\bar{U}(p_2,s_2)U(p_2',s_2')\bar{U}(p_2',s_2') \times U(p_1,s_1)\bar{U}(p_1,s_1)U(p_1',s_1')\bar{U}(p_1',s_1')U(p_2,s_2) \tag{6.8}$$

$$\mapsto \sum_{s_1,s_2}\sum_{s_1',s_2'}\overline{\mathcal{M}}_a\mathcal{M}_b = \frac{g^4}{(t-m_\pi^2)(u-m_\pi^2)} \times Tr[(\not{p}_2'+m_p)(\not{p}_1+m_p)(\not{p}_1'+m_p)(\not{p}_2+m_p)] \tag{6.9}$$

6.3 Trace formula of the gamma matrices

Let 1_4 is 4x4 identity matrix, we will have

$$Tr[1_4] = 4 \tag{6.10}$$
$$Tr[\gamma^\mu] = 0 \tag{6.11}$$
$$Tr[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu} \tag{6.12}$$
$$Tr[\gamma^\mu\gamma^\nu\gamma^\rho] = 0 \tag{6.13}$$
$$Tr[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4g^{\mu\nu}g^{\rho\sigma} - 4g^{\mu\rho}g^{\nu\sigma} + 4g^{\mu\sigma}g^{\nu\rho} \tag{6.14}$$
$$...$$
$$Tr[\gamma^{\mu_1}...\gamma^{\mu_n} = \sum_{k=2}^{n}(-1)^k g^{\mu_1\mu_k}Tr[\gamma^{\mu_2}...\gamma^{\mu_{k-1}}\gamma^{\mu_{k-2}}...\gamma^{\mu_n}] \tag{6.15}$$

After we have used the fact that $\{\gamma^\mu,\gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$ and $Tr[AB] = Tr[BA]$. For example

$$Tr[\gamma^a\gamma^b\gamma^c\gamma^d\gamma^e\gamma^f] = g^{ab}Tr[\gamma^c\gamma^d\gamma^e\gamma^f - g^{ac}Tr[\gamma^b\gamma^d\gamma^e\gamma^f]$$
$$+g^{ad}Tr[\gamma^b\gamma^c\gamma^e\gamma^f] - g^{ae}Tr[\gamma^b\gamma^c\gamma^d\gamma^f] + g^{af}Tr[\gamma^b\gamma^c\gamma^d\gamma^e]$$
$$= 4g^{ab}\left\{g^{cd}g^{ef} - g^{ce}g^{df} + g^{cf}g^{de}\right\}$$
$$-4g^{ac}\left\{g^{bd}g^{ef} - g^{be}g^{df} + g^{bf}g^{de}\right\}$$
$$+4g^{ad}\left\{g^{bc}g^{ef} - g^{be}g^{ce} + g^{bf}g^{ce}\right\}$$
$$-4g^{ae}\left\{g^{bc}g^{df} - g^{bd}g^{cf} + g^{bf}g^{cd}\right\}$$
$$+4g^{af}\left\{g^{bc}g^{de} - g^{bd}g^{ce} + g^{be}g^{cd}\right\}$$

And for short hand notation we always write

$$Tr[\not{p}\not{k}] = 4p\cdot k, \tag{6.16}$$
$$Tr[\not{p}\not{k}\not{p}'\not{k}'] = 4(p\cdot k)(p'\cdot k') - 4(p\cdot p')(k\cdot k') + 4(p\cdot k')(k\cdot p') \tag{6.17}$$

6.4 Differential cross section

From (5.51), (5.52) and (5.54) above we will have

$$\frac{1}{4}\sum_{s_1,s_2}\sum_{s_1',s_2'}\overline{\mathcal{M}}_a\mathcal{M}_a = \frac{4g^4}{(t-m_\pi^2)^2}(p_2'\cdot p_2 + m_p^2)(p_1'\cdot p_1 + m_p^2) \tag{6.18}$$

$$\frac{1}{4}\sum_{s_1,s_2}\sum_{s_1',s_2'}\overline{\mathcal{M}}_b\mathcal{M}_b = \frac{4g^4}{(u-m_\pi^2)^2}(p_2'\cdot p_1 + m_p^2)(p_1'\cdot p_2 + m_p^2) \tag{6.19}$$

$$\begin{aligned}\frac{1}{4}\sum_{s_1,s_2}\sum_{s_1',s_2'}2\overline{\mathcal{M}}_a\mathcal{M}_b &= \frac{2g^4}{(t-m_\pi^2)(u-m_\pi^2)}\\ \times\{[(p_2'\cdot p_1)(p_1'\cdot p_2) - (p_2'\cdot p_1')(p_1\cdot p_2) &+ (p_2'\cdot p_2)(p_1\cdot p_1')]\\ +m_p^2[(p_2'\cdot p_1) + (p_2'\cdot p_1') &+ (p_2'\cdot p_2)\\ +(p_1\cdot p_1') + (p_1\cdot p_2) &+ (p_1'\cdot p_2)] + m_p^4\}\end{aligned} \tag{6.20}$$

Since

$$s = (p_1+p_2)^2 = 2m_p^2 + 2p_1\cdot p_2 = (p_1'+p_2')^2 = 2m_p^2 + 2p_1'\cdot p_2'$$

$$\mapsto p_1\cdot p_2 = p_1'\cdot p_2' = \frac{s}{2} - m_p^2$$

$$t = (p_1-p_1')^2 = 2m_p^2 - 2p_1\cdot p_1' = (p_2-p_2')^2 = 2m_p^2 - 2p_2\cdot p_2'$$

$$\mapsto p_1\cdot p_1' = p_2\cdot p_2' = -(\frac{t}{2} - m_p^2)$$

$$u = (p_1-p_2')^2 = 2m_p^2 - 2p_1\cdot p_2' = (p_2-p_1')^2 = 2m_p^2 - 2p_2\cdot p_1'$$

$$\mapsto p_1\cdot p_2' = p_2\cdot p_1' = -(\frac{u}{2} - m_p^2)$$

We also have $s + t + u = 4m_p^2$

Then we have from above

$$\begin{aligned}\frac{1}{4}\sum_{s_1,s_2}\sum_{s_1',s_2'}\overline{\mathcal{M}}_a\mathcal{M}_a &= \frac{4g^2}{(t-m_\pi^2)^2}\left\{\left(\frac{t}{2} - m_p^2\right)^2 - 2m_p^2\left(\frac{t}{2} - m_p^2\right) + m_p^4\right\}\\ &= \frac{4g^2}{(t-m_\pi^2)^2}\left(\frac{t}{2} - 2m_p^2\right)^2 = \frac{g^4}{(t-m_\pi^2)^2}(t-4m_p^4)^2\end{aligned} \tag{6.21}$$

$$\begin{aligned}\frac{1}{4}\sum_{s_1,s_2}\sum_{s_1',s_2'}\overline{\mathcal{M}}_b\mathcal{M}_b &= \frac{4g^4}{(u-m_\pi^2)^2}\left\{\left(\frac{u}{2} - m_p^2\right)^2 - 2m_p^2\left(\frac{u}{2} - m_p^2\right) + m_p^4\right\}\\ &= \frac{4g^4}{(u-m_\pi^2)^2}\left(\frac{u}{2} - 2m_p^2\right)^2 = \frac{g^4}{(u-m_\pi^2)^2}(u-4m_p^2)^2\end{aligned} \tag{6.22}$$

$$
\begin{aligned}
\frac{1}{4}\sum_{s_1,s_2}\sum_{s_1',s_2'} 2\overline{\mathcal{M}}_a\mathcal{M}_b &= \frac{2g^4}{(t-m_\pi^2)(u-m_\pi^2)}\left\{\left(\frac{u}{2}-m_p^2\right)^2-\left(\frac{s}{2}-m_p^2\right)^2+\left(\frac{t}{2}-m_p^2\right)^2\right. \\
&\left. -2m_p^2\left[\left(\frac{u}{2}-m_p^2\right)-\left(\frac{s}{2}-m_p^2\right)+\left(\frac{t}{2}-m_p^2\right)\right]+m_p^4\right\} \\
&= \frac{2g^4}{(t-m_\pi^2)(u-m_\pi^2)}\left\{\left(\frac{t}{2}-2m_p^2\right)^2+\left(\frac{u}{2}-2m_p^2\right)^2-\left(\frac{s}{2}-2m_p^2\right)^2\right\} \\
&= \frac{g^4}{2(t-m_\pi^2)(u-m_\pi^2)}\left\{(t-4m_p^2)^2+(u-4m_p^2)^2-(s-4m_p^2)^2\right\} \\
&= \frac{g^4}{(t-m_\pi^2)(u-m_\pi^2)}(tu+4sm_p^2)
\end{aligned}
\tag{6.23}
$$

AFter we have used the fact that

$$
s^2 = (4m_p^2-t-u)^2 = t^2+u^2+(4m_p^2)^2-2t(4m_p^2)-2u(4m_p^2)+2tu
$$

Then

$$
|\overline{\mathcal{M}}|^2 = g^4\left\{\frac{(t-4m_p^2)^2}{(t-m_\pi^2)^2}+\frac{(u-4m_p^2)^2}{(u-m_\pi^2)^2}+\frac{(tu+4sm_p^2)}{(t-m_\pi^2)(u-m_\pi^2)}\right\} \tag{6.24}
$$

The differential cross section, from (4.30), is

$$
\frac{d\sigma}{dt} = \frac{1}{16\pi\lambda(s,m_p^2,m_p^2)}|\overline{\mathcal{M}}|^2 \tag{6.25}
$$

$$
\frac{d\sigma}{d\Omega} = \frac{1}{2\pi}2|\vec{p}||\vec{p}'|\frac{d\sigma}{dt} = \frac{1}{64\pi^2 s}|\overline{\mathcal{M}}|^2 \tag{6.26}
$$

For elastic scattering $|\vec{p}| = |\vec{p}'| = \frac{\lambda^{1/2}(s,m_p^2,m_p^2)}{2\sqrt{s}}$, see (4.17). Then we have from above

$$
\frac{d\sigma}{d\Omega} = \frac{g^4}{64\pi^2 s}\left\{\frac{(t-4m_p^2)^2}{(t-m_\pi^2)^2}+\frac{(u-4m_p^2)^2}{(u-m_\pi^2)^2}+\frac{(tu+4sm_p^2)}{(t-m_\pi^2)(u-m_\pi^2)}\right\} \tag{6.27}
$$