7 Vector Field Quantization

7.1 Vector field Lagrangian and Hamiltonian

For massless Maxwell vector field $A^\mu(x)$, its Lagrangian is written term of field strength tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \mapsto F^{0i} = -E^i,\ F^{ij} = \epsilon^{ijk} B^k,\ F^\mu = -F^{\nu\mu} \tag{7.1}$$

$$\mapsto \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2} \partial_\mu A_\nu F^{\mu\nu} \tag{7.2}$$

$$EOM \to \partial_\mu F^{\mu\nu} = \partial^2 A^\nu - \partial^\nu (\partial_\mu A^\mu) = 0 \tag{7.3}$$

This Lagrangian is invariant under gauge transformation $A^\mu \to A'^\mu(x) = A^\mu + \partial^\mu \chi(x)$ for any scalar function $\chi(x)$, and also the action $S[A^\mu]$ is invariant. Gauge fixing condition need to be applied for a physical vector field, the commonly used conditions are

- Lorentz condition: $\partial_\mu A^\mu(x) = 0$
- Coulomb condition: $\nabla \cdot \vec{A}(x) = 0$

Anyway they give the same result, so that for convenient we will work under Coulomb condition.

The conjugate momentum field is

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \partial_0 A_\mu} = -F^{0\mu} = -(\partial^0 A^\mu - \partial^\mu A^0) \mapsto F^{00} = 0,\ \pi^0 = 0 \tag{7.4}$$

This means that the temporal component of the vector field $A^0 = constant$ is not dynamical field, this results with zero momentum in this direction. For convenient we will set $A^0 = 0$, so that $A^\mu(x) = (0, \vec{A}(x))$ and $\pi^\mu = -\partial_0 A^\mu$. And the electric and magnetic fields can be calculated from its spatial part as

$$\vec{E}(x) = -\partial_0 \vec{A}(x),\ \vec{B}(x) = \nabla \times \vec{A}(x) \tag{7.5}$$

After Legendre transformation of the Lagrangian, we get vector field Hamiltonian in the form

$$\begin{aligned}\mathcal{H} = \pi^i \partial_0 A_i - \mathcal{L} &= -F^{0i} F_{0i} + \frac{1}{2} F^{0i} F_{0i} + \frac{1}{2} F^{ij} F_{ij} \\ &= \frac{1}{2}(E^i E^i + B^k B^k)\end{aligned} \tag{7.6}$$

$$H = \int d^3x \frac{1}{2} (\vec{E} \cdot \vec{E} + \vec{B} \cdot \vec{B}) \tag{7.7}$$

From its EOM, its trial free field solution is

$$A^\mu \sim a^\mu(k, \lambda) e^{-ik\cdot x} \mapsto k^2 a^\mu(k, \lambda) = 0 \to \omega^2 - \vec{k} \cdot \vec{k} = 0 \tag{7.8}$$

$$\omega^2 - \omega_k^2 = 0, \omega_k = |\vec{k}| \tag{7.9}$$

Then its general solution is written in term of Fourier transformation, with constraint condition of its dispersion, as

$$A^\mu(x) = \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi} \sum_\lambda \left(a^\mu(k,\lambda) e^{-ik\cdot x} + a^{*\mu}(k,\lambda) e^{ik\cdot x} \right) \times (2\pi)\delta(\omega^2 - \omega_k^2)\theta(\omega) \tag{7.10}$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_\lambda \left(a^\mu(k,\lambda) e^{-ik\cdot x} + a^{*\mu}(k,\lambda) e^{ik\cdot x} \right)_{\omega=\omega_k} \tag{7.11}$$

After we have used the identity of the delta function. Let us assign the polarization tensor

$$a^\mu(k,\lambda) = \epsilon^\mu(k,\lambda) a(k) \mapsto \sum_\lambda \epsilon^\mu(k,\lambda)\epsilon^\nu(k',\lambda) = -g^{\mu\nu}\delta_{k,k'} \tag{7.12}$$

From Coulomb condition, we will have $\epsilon^\mu = (0, \hat{\epsilon})$, and $\vec{k} \cdot \hat{\epsilon} = 0$.

Its conjugate momentum field, $\pi^\mu = (0, \vec{\pi}(x))$, will appear in the form

$$\vec{\pi}(x) = -\partial_0 \vec{A}(x) = \vec{E}(x) = \frac{i}{2} \int \frac{d^3k}{(2\pi)^3} \sum_\lambda \hat{\epsilon}(k,\lambda) \left\{ a(k,\lambda) e^{-ik\cdot x} - a^*(k,\lambda) e^{ik\cdot x} \right\}_{\omega=\omega_k} \tag{7.13}$$

with $\pi^0 = 0, \pi^i(x) = E^i(x)$. We also have

$$\nabla \cdot \vec{A}(x) = i \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_\lambda \vec{k} \cdot \hat{\epsilon}(k,\lambda) \left[a(k,\lambda) e^{-ik\cdot x} - a^*(k,\lambda) e^{ik\cdot x} \right]_{\omega=\omega_k} \tag{7.14}$$

$$\vec{B}(x) = \nabla \times \vec{A}(x) = i \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_\lambda (\vec{k} \times \hat{\epsilon}(k,\lambda)) \left[a(k,\lambda) e^{-ik\cdot x} - a^*(k,\lambda) e^{ik\cdot x} \right]_{\omega=\omega_k} \tag{7.15}$$

7.2 Canonical quantization

We promote the vector field and its conjugate momentum field to be field operators, satisfy *an equal time commutation relation*

$$[A^\mu(x), \pi^\nu(y)]_{x^0=y^0} = -ig^{\mu\nu}\delta^{(3)}(\vec{x} - \vec{y}) \tag{7.16}$$

From free field solution, let us determine

$$[A^\mu(x),\pi^\nu(y)]_{x^0=y^0=0} = \frac{i}{2}\int\frac{d^3k}{(2\pi)^3 2\omega_k}\int\frac{d^3k'}{(2\pi)^3}\sum_{\lambda,\lambda'}\epsilon^\mu(k,\lambda)\epsilon^\nu(k',\lambda')$$

$$\times\Big[\Big(a(k,\lambda)e^{i\vec{k}\cdot\vec{x}}+a^\dagger(k,\lambda)e^{-i\vec{k}\cdot\vec{x}}\Big),\Big(a(k',\lambda')e^{i\vec{k}'\cdot\vec{y}}-a^\dagger(k'\lambda')e^{-i\vec{k}'\cdot\vec{y}}\Big)\Big]$$

$$= \frac{i}{2}\int\frac{d^3k}{(2\pi)^3 2\omega_k}\int\frac{d^3k'}{(2\pi)^3}\sum_{\lambda,\lambda'}\epsilon^\mu(k,\lambda)\epsilon^\nu(k',\lambda')$$

$$\times\Big\{[a(k,\lambda),a(k',\lambda')]e^{i\vec{k}\cdot\vec{x}+i\vec{k}'\cdot\vec{y}}-[a(k,\lambda),a^\dagger(k',\lambda')]e^{i\vec{k}\cdot\vec{x}-i\vec{k}'\cdot\vec{y}}$$

$$+[a^\dagger(k,\lambda),a(k',\lambda')]e^{-i\vec{k}\cdot\vec{x}+i\vec{k}'\cdot\vec{y}}-[a^\dagger(k,\lambda),a^\dagger(k',\lambda')]e^{-i\vec{k}\cdot\vec{x}-i\vec{k}'\cdot\vec{y}}\Big\} \tag{7.17}$$

Let us assign the following commutation relations

$$[a(k,\lambda),a(k',\lambda')]=0=[a^\dagger(k,\lambda),a^\dagger(k',\lambda')] \tag{7.18}$$

$$\big[a(k,\lambda),a^\dagger(k',\lambda')\big]=(2\pi)^3 2\omega_k\delta_{\lambda\lambda'}\delta^{(3)}(\vec{k}-\vec{k}') \tag{7.19}$$

From above we have

$$[A^\mu(x),\pi^\nu(y)]_{x^0=y^0=0} = -\frac{i}{2}g^{\mu\nu}\int\frac{d^3k}{(2\pi)^3}\Big\{e^{i\vec{k}\cdot(\vec{x}-\vec{y})}+e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}\Big\}$$

$$= -ig^{\mu\nu}\delta^{(3)}(\vec{x}-\vec{y}) \tag{7.20}$$

After we have changed sign $\vec{k}\to-\vec{k}$ on the first term.

Next let us calculate the Hamiltonian operator, at $x^0=y^0=0$, as

$$H = \frac{1}{2}\int d^3x\Big(\vec{E}(x)\cdot\vec{E}(x)+\vec{B}(x)\vec{B}(x)\Big)$$

$$= \frac{1}{2}\int d^3x\int\frac{d^3k}{(2\pi)^3 2\omega_k}\int\frac{d^3k'}{(2\pi)^3 2\omega_{k'}}\sum_{\lambda,\lambda'}$$

$$\times\Big\{-\omega_k\omega_{k'}\hat{\epsilon}(k,\lambda)\cdot\hat{\epsilon}(k',\lambda')[(a(k,\lambda)e^{i\vec{k}\cdot x}-a^\dagger(k,\lambda)e^{-i\vec{k}\cdot\vec{x}}]$$

$$\times[a(k',\lambda')e^{i\vec{k}'\cdot\vec{x}}-a^\dagger(k',\lambda')e^{-i\vec{k}'\cdot\vec{x}}]$$

$$-(\vec{k}\times\hat{\epsilon}(k,\lambda))\cdot(\vec{k}'\times\hat{\epsilon}(k',\lambda'))[(a(k,\lambda)e^{i\vec{k}\cdot x}+a^\dagger(k,\lambda)e^{-i\vec{k}\cdot\vec{x}}]$$

$$\times[a(k',\lambda')e^{i\vec{k}'\cdot\vec{x}}+a^\dagger(k',\lambda')e^{-i\vec{k}'\cdot\vec{x}}]\Big\}$$

$$= \frac{1}{2}\int d^3x\int\frac{d^3k}{(2\pi)^3 2\omega_k}\int\frac{d^3k'}{(2\pi)^3 2\omega_{k'}}\sum_{\lambda,\lambda'}\hat{\epsilon}(k,\lambda)\cdot\hat{\epsilon}(k',\lambda')$$

$$\times\Big\{-\omega_k\omega_{k'}\Big(a(k,\lambda)a(k',\lambda')e^{i(\vec{k}+\vec{k}')\cdot\vec{x}}-a(k,\lambda)a^\dagger(k',\lambda')e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}$$

$$-a^\dagger(k,\lambda)a(k',\lambda')e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}}+a^\dagger(k,\lambda)a^\dagger(k',\lambda')e^{-i(\vec{k}+\vec{k}')\cdot\vec{x}}\Big)$$

$$-\vec{k}\cdot\vec{k}'\Big(a(k,\lambda)a(k',\lambda')e^{i(\vec{k}+\vec{k})\cdot\vec{x}}-a(k,\lambda)a^\dagger(k',\lambda')e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}$$

$$-a^\dagger(k,\lambda)a(k',\lambda')e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}}+a^\dagger(k,\lambda)a^\dagger(k',\lambda')e^{-i(\vec{k}+\vec{k}')\cdot\vec{x}}\Big)\Big\} \tag{7.21}$$

After using identity $[\vec{k}\times\hat{\epsilon}(k,\lambda)]\cdot[\vec{k}'\times\hat{\epsilon}(k',\lambda')] = [\vec{k}\cdot\vec{k}'][\hat{\epsilon}(k,\lambda)\cdot\hat{\epsilon}(k',\lambda')]$, under Coulomb condition. Integrate $\int d^3x$ we will get delta functions. Integrate d^3k' using delta function, using orthogonality of polarization $\hat{\epsilon}(k,\lambda)$ and sum overall polarization λ', and using the fact that $\omega_k = |\vec{k}|$. We have cancellation terms when $\vec{k}' = -\vec{k}$, we have contribution terms when $\vec{k}' = \vec{k}$. Finally we have

$$H = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2\omega_k}\omega_k \sum_\lambda [a(k,\lambda)a^\dagger(k,\lambda) + a^\dagger(k,\lambda)a(k,\lambda)] \tag{7.22}$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\omega_k \sum_\lambda a^\dagger(k,\lambda)a(k,\lambda) + 2\cdot\frac{1}{2}\int d^3k\delta^{(3)}(0) \tag{7.23}$$

where the last term is an infinite vacuum energy, without particle. We can get rid of this energy by defining *normal ordering* of the Hamiltonian, and define the vacuum state $|o\rangle$ in which particle can be created from and destroyed into as

$$a(k,\lambda)|0\rangle = 0,\ a^\dagger(k,\lambda)|0\rangle = |k,\lambda\rangle \tag{7.24}$$

$$: H := \int \frac{d^3k}{(2\pi)^3 2\omega_k}\omega_k \sum_\lambda a^\dagger(k,\lambda)a(k,\lambda) \tag{7.25}$$

This shows that quantum of vector field is an infinite set of bosonic harmonic oscillators.

7.3 Vector field propagator

Feynman propagator of the vector field is

$$\begin{aligned}\Delta^{\mu\nu}(x-y) = \langle 0|T[A^\mu(x)A^\nu(y)]|0\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\int \frac{d^3k'}{(2\pi)^3 2\omega_{k'}}\sum_{\lambda,\lambda'} \\ \epsilon^\mu(k,\lambda)\epsilon^\nu(k',\lambda')\Big\{\theta(x^0-y^0)\langle 0|a(k,\lambda)a^\dagger(k',\lambda')|0\rangle e^{-ik\cdot x+ik'\cdot y} \\ +\theta(y^0-x^0)\langle 0|a(k',\lambda')a^\dagger(k,\lambda)|0\rangle e^{-ik'\cdot y+ik\cdot x}\Big\} \\ = -g^{\mu\nu}\int \frac{d^3k}{(2\pi)^3 2\omega_k}\Big[\theta(x^0-y^0)e^{-ik\cdot(x-y)} + \theta(y^0-x^0)e^{ik\cdot(x-y)}\Big] \\ = -g^{\mu\nu}\int \frac{d^3k}{(2\pi)^3}\int \frac{d\omega}{2\pi}\frac{i}{k^2+i\epsilon}e^{ik\cdot(x-y)}\end{aligned} \tag{7.26}$$

This means that

$$\Delta^{\mu\nu}(k) = \frac{-ig^{\mu\nu}}{k^2+i\epsilon} \tag{7.27}$$

7.4 LSZ reduction formula

From free vector field solution and its cojugate momentum field, we can have

$$\begin{aligned}\int d^3x e^{-i\vec{k}\cdot\vec{x}}\hat{\epsilon}(k,\lambda)\vec{A}(x) &= \int \frac{d^3k'}{(2\pi)^3 2\omega_{k'}}\sum_{\lambda'}\hat{\epsilon}(k,\lambda)\hat{\epsilon}(k',\lambda') \\ &\times[a(k',\lambda')e^{-i\omega_{k'}x^0}\int d^3x e^{i(\vec{k}'-\vec{k})\cdot\vec{x}} \\ &+a^\dagger(k',\lambda')e^{i\omega_{k'}x^0}\int d^3x e^{-i(\vec{k}'+\vec{k})\cdot\vec{x}}] \\ &= \frac{1}{2\omega_k}[a(k,\lambda)e^{-i\omega_k x^0} + a^\dagger(-k,\lambda)e^{i\omega_k x^0}] \qquad (7.28)\end{aligned}$$

$$\begin{aligned}\int d^3x e^{-i\vec{k}\cdot\vec{x}}\hat{\epsilon}(k,\lambda)(-\partial_0\vec{A}(x) &= \frac{i}{2}\int \frac{d^3k'}{(2\pi)^3}\sum_{\lambda'}\hat{\epsilon}(k,\lambda)\hat{\epsilon}(k',\lambda') \\ &\times[a(k',\lambda')e^{-i\omega_{k'}x^0}\int d^3x e^{i(\vec{k}'-\vec{k})\cdot\vec{x}} \\ &-a^\dagger(k',\lambda')e^{i\omega_{k'}x^0}\int d^3x e^{-i(\vec{k}'+\vec{k})\cdot\vec{x}}] \\ &= \frac{i}{2}[a(k,\lambda)e^{-i\omega_k x^0} - a^\dagger(-k,\lambda)e^{i\omega_k x^0}] \qquad (7.29)\end{aligned}$$

Then we have

$$\begin{aligned}a(k,\lambda) = \int d^3x e^{ik\cdot x}\hat{\epsilon}(k,\lambda)[\omega_k + i\partial_0]\vec{A}(x) &\equiv \int d^3x e^{ik\cdot x}\hat{\epsilon}(k,\lambda) i\overleftrightarrow{\partial}_0\vec{A}(x) \\ &= \int d^4x \partial_0\left\{e^{ik\cdot x}\hat{\epsilon}(k,\lambda)i\overleftrightarrow{\partial}_0\vec{A}(x)\right\} \\ = \int d^3x\hat{\epsilon}(k,\lambda)&\left\{(-i\partial_0^2 e^{ik\cdot x})\vec{A}(x) + e^{ik\cdot x}i\partial_0^2\vec{A}(x)\right\}\end{aligned}$$

$$\mapsto a(k,\lambda) = i\int d^3x e^{ik\cdot x}\hat{\epsilon}(k,\lambda)\partial_x^2\vec{A}(x) \qquad (7.30)$$

After we have used vector field EOM

$$\partial_x^2 e^{ik\cdot x} = (\partial_0^2 - \nabla_x^2)e^{ik\cdot x} = 0 \mapsto \partial_0^2 e^{ik\cdot x} = \nabla_x^2 e^{ik\cdot x}$$

Its conjugation is

$$a^\dagger(k,\lambda) = -i\int d^4x e^{-ik\cdot x}\hat{\epsilon}(k,\lambda)\partial_x^2\vec{A}(x) \qquad (7.31)$$

The incoming state of a single photon is

$$|k,\lambda\rangle = a^\dagger(k,\lambda)|0\rangle = -i\int d^4x e^{-ik\cdot x}\hat{\epsilon}(k,\lambda)\partial_x^2\vec{A}(x)|0\rangle \qquad (7.32)$$

$$\langle k,\lambda| = \langle 0|a(k,\lambda) = i\int d^4x e^{ik\cdot x}\hat{\epsilon}(k,\lambda)\partial_x^2\langle 0|\vec{A}(x) \qquad (7.33)$$

7.5 Scalar quantum electrodynamics (ScQED)

The model Lagrangian of ScQED is

$$\mathcal{L} = \partial_\mu \phi^*(x)\partial^\mu \phi(x) - m^2\phi^*(x)\phi(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - qj_\mu(x)A^\mu(x) \tag{7.34}$$

$$j_\mu(x) = -i[\phi^*(x)\partial_\mu\phi(x) - \phi(x)\partial_\mu\phi^*(x)] \tag{7.35}$$

where $j_\mu(x)$ is complex scalar current density.

ScQED Feynman rules can be assigned as in the following, See figure (6.1).

- scalar propagator is $\Delta_s(p) = \frac{-i}{p^2-m^2+i\epsilon}$
- photon propagator is $\Delta^{\mu\nu}(p) = \frac{-ig^{\mu\nu}}{p^2+i\epsilon}$
- interaction vertex is $-ie(p-p')^\mu$, where p^μ is incoming scalar and p'^μ is outgoing scalar momenta
- incoming and outgoing (real) photon polarization $\epsilon^\mu(p,\lambda)$
- symmetry factor is 1 at each vertex.

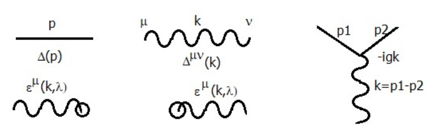

Figure 7.1: ScQED Feynman rules.

Let us determine the Compton scattering in ScQED, see figure (6.2)

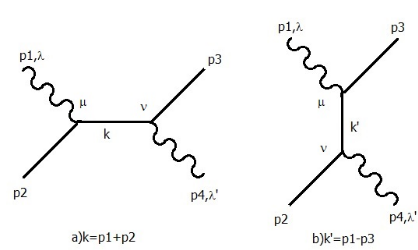

Figure 7.2: Tree diagrams of Compton scattering in ScQED.

The corresponding amplitudes are

$$\mathcal{M}_a = -e^2\epsilon^\mu(p_1,\lambda_1)\epsilon^\nu(q_1,\lambda_2)p_{1\mu}\frac{i}{(p_1+p_2)^2-m^2}p_{4\nu} = -ie^2\frac{p_1\cdot p_4}{s-m^2}\delta_{\lambda_1\lambda_2} \tag{7.36}$$

$$\mathcal{M}_b = -e^2\epsilon^\mu(p_1,\lambda_1)\epsilon^\nu(q_2,\lambda_2)p_{1\mu}\frac{i}{(p_1-p_3)^2-m^2}p_{4\nu} = -ie^2\frac{p_1\cdot p_4}{t-m^2}\delta_{\lambda_1\lambda_2} \tag{7.37}$$

The amplitude squared will be determined with sum overall outgoing polarization and averaged overall incoming polarization as

$$
\begin{aligned}
\overline{|\mathcal{M}|^2} &= \frac{1}{2}\sum_{\lambda_1,\lambda_2} |\mathcal{M}_a + \mathcal{M}_b|^2 \\
&= \frac{e^4}{2}(p_1 \cdot p_4)^2 \left\{ \frac{1}{(s-m^2)^2} + \frac{2}{(s-m^2)(t-m^2)} + \frac{1}{(t-m^2)^2} \right\} \\
&= \frac{e^4}{8}\frac{u^2}{4}\left(\frac{1}{s-m^2} + \frac{1}{t-m^2}\right)^2 = \frac{e^4}{32}\frac{(2m^2-s-t)^4}{(s-m^2)^2(t-m^2)^2} \qquad (7.38)
\end{aligned}
$$