8 Spinor Quantum Electrodynamics

8.1 Lagrangian and Feynman rules

Spinor quantum electrodynamics describes electron-electron, electron-position interaction via photon exchange, or electron-photon interaction. Its Lagrangian is

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\bar{\psi}\gamma^{\mu}\psi A_{\mu} \tag{8.1}$$

Its Feynman rules, see figure (8.1), are:

- electron propagator is $\Delta^{(-)}(p) = \frac{i(p\!\!\!/+m)}{p^2-m^2+i\epsilon}$
- positron propagator is $\Delta^{(+)}(p) = \frac{i(p\!\!\!/-m)}{p^2-m^2+i\epsilon}$
- photon propagator, in Lorentz gauge, is $\Delta^{\mu\nu}(k) = \frac{-ig^{\mu\nu}}{k^2+i\epsilon}$
- electron/positron-photon interaction vertex is $-ie\gamma^{\mu}$
- incoming electron spinor is $U(p,s)$, and out going electron spinor is $\bar{U}(p,s)$
- incoming positron spinor is $V(p,s)$, and out going positron spinor is $\bar{V}(p,s)$
- incoming/outgoing photon polarization is $\epsilon^{\mu}(k,\lambda)$
- symmetry factor at each vertex is 1

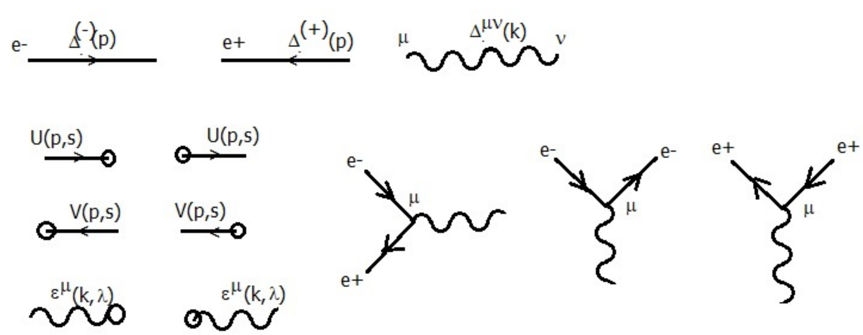

Figure 8.1: QED Feynman rules.

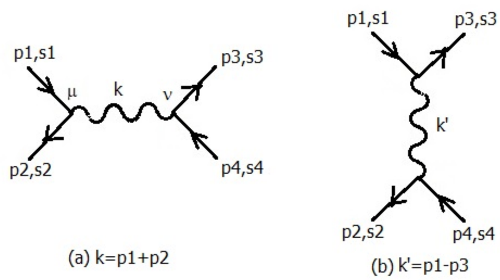

Figure 8.2: Bhabha scattering diagrams.

8.2 Bhabha scattering

Bhabha scattering is $e^-e^+ \to e^-e^+$ process, its corresponding diagrams appear in figure (8.2). Their corresponding amplitudes are

$$\begin{aligned}\mathcal{M}_a &= -e^2 \bar{V}(2)\gamma^\mu U(1)\frac{-ig_{\mu\nu}}{k^2}\bar{U}(3)\gamma^\nu V(4) \\ &= i\frac{e^2}{s}[\bar{V}(2)\gamma^\mu U(1)][\bar{U}(3)\gamma_\mu V(4)] \end{aligned} \quad (8.2)$$

$$\begin{aligned}\mathcal{M}_b &= -e^2 \bar{U}(3)\gamma^\mu U(1)\frac{-ig_{\mu\nu}}{k'^2}\bar{V}(2)\gamma^\nu V(4) \\ & i\frac{e^2}{t}[\bar{U}(3)\gamma^\mu U(1)][\bar{V}(2)\gamma_\mu V(4)] \end{aligned} \quad (8.3)$$

The averaged amplitude squared is determined in the form

$$\overline{|\mathcal{M}|^2} = \frac{1}{4}\sum_{s_1,s_2}\sum_{s_3,s_4}|\mathcal{M}_a + \mathcal{M}_b|^2 = \overline{|\mathcal{M}_2|}^2 + \overline{|\mathcal{M}_b|}^2 + 2\overline{|\mathcal{M}_a^*\mathcal{M}_b|} \quad (8.4)$$

where

$$|\mathcal{M}_a|^2 = \frac{e^4}{s^2}[\bar{U}(1)\gamma^\mu V(2)][\bar{V}(2)\gamma^\nu U(1)][\bar{V}(4)\gamma_\mu U(3)][\bar{U}(3)\gamma_\nu V(4)] \quad (8.5)$$

$$\begin{aligned}\mapsto \overline{|\mathcal{M}_a|^2} &= \frac{e^4}{s^2}Tr[\gamma^\mu(\not{p}_2 - m)\gamma^\nu(\not{p}_1 + m)] \\ &\times Tr[\gamma_\mu(\not{p}_3 + m)\gamma_\nu(\not{p}_4 - m)] \quad (8.6) \\ &= \frac{8e^4}{s^2}\{8m^4 + 4m^2(s - t + u) + t^2 + u^2\} \quad (8.7)\end{aligned}$$

$$|\mathcal{M}_b|^2 = \frac{e^4}{t^2}[\bar{U}(1)\gamma^\mu U(3)][\bar{U}(3)\gamma^\nu U(1)][\bar{V}(4)\gamma_\mu V(2)][\bar{V}(2)\gamma_\nu V(4)] \tag{8.8}$$

$$\mapsto \overline{|\mathcal{M}_b|^2} = \frac{e^4}{t^2}Tr[\gamma^\mu(\not{p}_3 + m)\gamma^\nu(\not{p}_1 + m)] \times Tr[\gamma_\mu(\not{p}_2 - m)\gamma_\nu(\not{p}_4 - m)] \tag{8.9}$$

$$= \frac{8e^4}{t^2}\{8m^2 - 4m^2(s - t + u) + s^2 + u^2\} \tag{8.10}$$

$$|\mathcal{M}_a^*\mathcal{M}_b| = \frac{e^4}{st}[\bar{V}(4)\gamma_\mu U(3)][\bar{U}(3)\gamma^\nu U(1)][\bar{U}(1)\gamma^\mu V(2)][\bar{V}(2)\gamma_\nu V(4)] \tag{8.11}$$

$$\mapsto 2\overline{|\mathcal{M}_a^*\mathcal{M}_b|^2} = \frac{2e^4}{st}Tr[\gamma_\mu(\not{p}_3 + m)\gamma^\nu(\not{p}_1 + m)\gamma^\mu(\not{p}_2 - m)\gamma_\nu(\not{p}_4 - m)] \tag{8.12}$$

$$= \frac{16e^4}{st}\{4m^4 + 2m^2(s + t - 3u) + u^2\} \tag{8.13}$$

Finally we have

$$\overline{|\mathcal{M}|^2} = 8e^4 \left\{ \frac{8m^4 - 4m^2(s - t - u) + t^2 + u^2}{s^2} + \frac{8m^4 - 4m^2(s - t + u) + s^2 + u^2}{t^2} - \frac{2(4m^2 + 2m^2(s + t) - 6m^2u + u^2}{st} \right\} \tag{8.14}$$

In the ultra-relativistic limit, we have

$$\overline{|\mathcal{M}|^2} \xrightarrow{m \to 0} 8e^4 \left\{ \frac{s^2 + u^2}{t^2} - \frac{2u^2}{st} + \frac{t^2 + u^2}{s^2} \right\} \tag{8.15}$$

See FeynCalc code at *bhabha.nb*.

8.3 Compton scattering

Compton scattering is $\gamma e^- \to \gamma e^-$ process. Its corresponding diagrams appear in figure (8.3).
Their corresponding amplitudes are

$$\mathcal{M}_a = -e^2\epsilon^\mu(p_1, \lambda)\epsilon^\nu(p_4, \lambda')\bar{U}(p_3, s')\gamma_\nu \frac{i(\not{k} + m)}{k^2 - m^2}\gamma_\mu U(p_2, s)$$
$$= -i\frac{e^2}{s - m^2}\epsilon^\mu(p_1, \lambda)\epsilon^\nu(p_4, \lambda')\bar{U}(p_3, s')\gamma_\nu(\not{p}_1 + \not{p}_2 + m)\gamma_\mu U(p_2, s) \tag{8.16}$$

$$\mathcal{M}_b = -e^2\epsilon^\mu(p_1, \lambda)\epsilon^\nu(p_4, \lambda')\bar{U}(p_3, s')\gamma_\mu \frac{i(\not{k}' + m)}{k'^2 - m^2}\gamma_\nu U(p_2, s)$$
$$= -i\frac{e^2}{t - m^2}\epsilon^\mu(p_1, \lambda)\epsilon^\nu(p_4, \lambda')\bar{U}(p_3, s')\gamma_\mu(\not{p}_1 - \not{p}_3 + m)\gamma_\nu U(p_2, s) \tag{8.17}$$

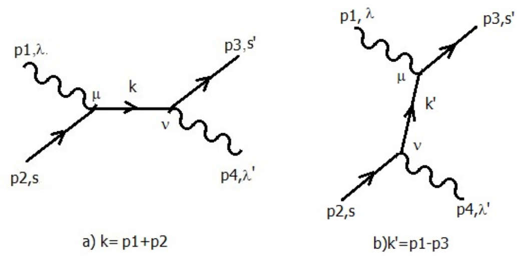

Figure 8.3: Compton scattering diagrams.

The averaged amplitude squared is calculated as

$$\overline{|\mathcal{M}|^2} = \frac{1}{4}\sum_{s,\lambda}\sum_{s',\lambda'}|\mathcal{M}_a + \mathcal{M}_b|^2 = \overline{|\mathcal{M}_a|^2} + \overline{|\mathcal{M}_b|^2} + 2\overline{|\mathcal{M}_a^*\mathcal{M}_b|^2} \tag{8.18}$$

where

$$\begin{aligned} |\mathcal{M}_a|^2 = & \frac{e^4}{(s-m^2)^2}[\epsilon^\mu(p_1,\lambda)\epsilon^\rho(p_1,s)][\epsilon^\nu(p_4,\lambda')\epsilon^\sigma(p_4,\lambda')] \\ & \times[\bar{U}(p_2,s)\gamma_\rho(\not{p}_1+\not{p}_2+m)\gamma_\sigma U(p_3,s')] \\ & \times[\bar{U}(p_3,s')\gamma_\nu(\not{p}_1+\not{p}_2+m)\gamma_\mu U(p_2,s)] \end{aligned} \tag{8.19}$$

$$\begin{aligned} \mapsto \overline{|\mathcal{M}_a|^2} = & \frac{1}{4}\frac{e^4}{(s-m^2)^2}Tr[\gamma^\mu(\not{p}_1+\not{p}_2+m)\gamma^\nu(\not{p}_3+m) \\ & \times\gamma_\nu(\not{p}_1+\not{p}_2+m)\gamma_\mu(\not{p}_2+m)] \end{aligned} \tag{8.20}$$

$$= \frac{8e^4}{(s-m^2)^2}\{6m^4 - m^2(6s+4t+3u) + st\} \tag{8.21}$$

$$\begin{aligned} |\mathcal{M}_b|^2 = & \frac{e^4}{(t-m^2)^2}[\epsilon^\mu(p_1,\lambda)\epsilon^\rho(p_1,\lambda)][\epsilon^\nu(p_4,\lambda')\epsilon^\sigma(p_4,\lambda')] \\ & \times[\bar{U}(p_2,s)\gamma_\sigma(\not{p}_1-\not{p}_3+m)\gamma_\rho U(p_3,s')] \\ & \times[\bar{U}(p_3,s')\gamma_\mu(\not{p}_1-\not{p}_3+m)\gamma_\nu U(p_2,s)] \end{aligned} \tag{8.22}$$

$$\begin{aligned} \mapsto \overline{|\mathcal{M}_b|^2} = & \frac{1}{4}\frac{e^4}{(t-m^2)^2}Tr[\gamma^\nu(\not{p}_1-\not{p}_3+m)\gamma^\mu(\not{p}_3+m) \\ & \times\gamma_\mu(\not{p}_1-\not{p}_3+m)\gamma_\nu(\not{P}_2+m)] \end{aligned} \tag{8.23}$$

$$= \frac{8e^4}{(t-m^2)^2}\{26m^4 + m^2(-4s+14t-5u) - st\} \tag{8.24}$$

$$|\mathcal{M}_a^*\mathcal{M}_b| = \frac{e^4}{(s-m^2)(t-m^2)}[\epsilon^\mu(p_1,\lambda)\epsilon^\rho(p_1,\lambda)][\epsilon^\nu(p_3,\lambda')\epsilon^\sigma(p_3,\lambda')]$$
$$\times[\bar{U}(p_2,s)\gamma_\rho(\not{p}_1+\not{p}_2+m)\gamma_\sigma U(p_3,s')]$$
$$\times[\bar{U}(p_3,s')\gamma_\mu(\not{p}_1-\not{p}_3+m)\gamma_\nu U(p_2,s)] \quad (8.25)$$
$$\mapsto 2\overline{|\mathcal{M}_a^*\mathcal{M}_b|} = \frac{e^4}{(s-m^2)(t-m^2)}Tr[\gamma^\mu(\not{p}_1+\not{p}_2+m)\gamma^\nu(\not{p}_3+m)...$$
$$...\gamma_\mu(\not{p}_1-\not{p}_3+m)\gamma_\nu(\not{p}_2+m)] \quad (8.26)$$
$$= \frac{16e^4}{(s-m^2)(t-m^2)}\{2m^4+m^2(s-3t-6u)+u(s+t+u)\} \quad (8.27)$$

Finally we have

$$\overline{|\mathcal{M}|^2} = 8e^4\left\{\frac{6m^4-m^2(6s+4t+3u)+s}{(s-m^2)^2}\right.$$
$$+\frac{26m^4+m^2(-4s+14t-5u)-s}{(t-m^2)^2}$$
$$\left.+\frac{2(2m^4+m^2(s-3t-6u)+u(s+t+u))}{(s-m^2)(t-m^2)}\right\} \quad (8.28)$$

At ultra-relativistic limit, we will have

$$\overline{|\mathcal{M}|^2} \xrightarrow{m\to 0} 8e^4\left\{\frac{2u(s+t+u)}{st}-\frac{s}{t}+\frac{t}{s}\right\} \quad (8.29)$$

See FeynCalc code at *compton.nb*.

8.4 Pair creation

The pair creation is $\gamma\gamma \to e^-e^+$ process. Its corresponding diagrams appear in figure (8.4).

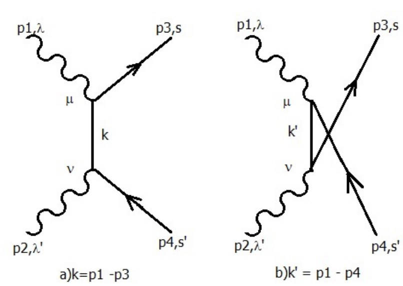

Figure 8.4: Pair creation diagrams.

Their corresponding amplitudes are

$$\mathcal{M}_a = -e^2\epsilon^\mu(p_1,\lambda)\epsilon^\nu(p_2,\lambda')\bar{U}(p_3,s)\gamma_\mu\frac{i(\not{k}+m)}{k^2-m^2}\gamma_\nu V(p_4,s')$$
$$= -i\frac{e^2}{t-m^2}\epsilon^\mu(p_1,\lambda)\epsilon^\nu(p_2,\lambda')\bar{U}(p_3,s)\gamma_\mu(\not{p}_1-\not{p}_3+m)\gamma_\nu V(p_4,s') \tag{8.30}$$

$$\mathcal{M}_b = -e^2\epsilon^\mu(p_1,\lambda)\epsilon^\nu(p_2,\lambda')\bar{U}(p_3,s)\gamma_\nu\frac{i(\not{k}'+m)}{k'^2-m^2}\gamma_\mu V(p_4,s')$$
$$= -i\frac{e^2}{u-m^2}\epsilon^\mu(p_1,\lambda)\epsilon^\nu(p_2,\lambda')\bar{U}(p_3,s)\gamma_\nu(\not{p}_1-\not{p}_4+m)\gamma_\mu V(p_4,s') \tag{8.31}$$

The averaged amplitude squared is written in the form

$$\overline{|\mathcal{M}|^2} = \frac{1}{4}\sum_{\lambda,\lambda'}\sum_{s,s'}|\mathcal{M}_a+\mathcal{M}_b|^2 = \overline{|\mathcal{M}_a|^2}+\overline{|\mathcal{M}_b|^2}+2\overline{|\mathcal{M}_a^*\mathcal{M}_b|} \tag{8.32}$$

where

$$|\mathcal{M}_a|^2 = \frac{e^4}{(t-m^2)^2}[\epsilon^\mu(p_1,\lambda)\epsilon^\rho(p_1,\lambda)][\epsilon^\nu(p_2,\lambda')\epsilon^\sigma(p_2,\lambda')]$$
$$\times\bar{V}(p_4,s')\gamma_\sigma(\not{p}_1-\not{p}_3+m)\gamma_\rho U(p_3,s)$$
$$\times\bar{U}(p_3,s)\gamma_\mu(\not{p}_1-\not{p}_3+m)\gamma_\nu V(p_4,s') \tag{8.33}$$
$$\mapsto \overline{|\mathcal{M}_a|^2} = \frac{e^4}{(t-m^2)^2}Tr[\sigma^\nu(\not{p}_1-\not{p}_3+m)\gamma^\mu(\not{p}_3+m)$$
$$\times\gamma_\mu(\not{p}_1-\not{p}_3+m)\gamma_\nu(\not{p}_4-m)] \tag{8.34}$$
$$= -\frac{8e^4}{(t-m^2)^2}\{13m^4+m^2(-5s+12t+4u)-tu\} \tag{8.35}$$

$$|\mathcal{M}_b|^2 = \frac{e^4}{(u-m^2)^2}[\epsilon^\mu(p_1,\lambda)\epsilon^\rho(p_1,\lambda)][\epsilon^\nu(p_2,\lambda')\epsilon^\sigma(p_2,\lambda')]$$
$$\times\bar{V}(p_4,s')\gamma_\rho(\not{p}_1-\not{p}_4+m)\gamma_\sigma U(p_3,s)$$
$$\times\bar{U}(p_3,s)\gamma_\nu(\not{p}_1-\not{p}_4+m)\gamma_\mu V(p_4,s') \tag{8.36}$$
$$\mapsto \overline{|\mathcal{M}_b|^2} = \frac{e^4}{(u-m^2)^2}Tr[\gamma^\mu(\not{p}_1-\not{p}_4+m)\gamma^\nu(\not{p}_3+m)...$$
$$...\gamma_\nu(\not{p}_1-\not{p}_4+m)\gamma_\mu(\not{p}_4-m)] \tag{8.37}$$
$$= -\frac{8e^4}{(u-m^2)^2}\{m^4+m^2(3s+4t+5u)-tu\} \tag{8.38}$$

$$|\mathcal{M}_a^*\mathcal{M}_b| = \frac{e^4}{(t-m^2)(u-m^2)}[\epsilon^\mu(p_1,\lambda)\epsilon^\rho(p_1,\lambda)][\epsilon^\nu(p_2,\lambda')\epsilon^\sigma(p_2,\lambda')]$$
$$\times[\bar{V}(p_4,s')\gamma_\sigma(\not{p}_1-\not{p}_3+m)\gamma_\rho U(p_3,s)...$$
$$\times...\bar{U}(p_3,s)\gamma_\nu(\not{p}_1-\not{p}_4+m)\gamma_\mu V(p_4,s')] \quad (8.39)$$
$$\mapsto 2\overline{|\mathcal{M}_a^*\mathcal{M}_b|} = \frac{2e^4}{(t-m^2)(u-m^2)}Tr[\gamma^\nu(\not{p}_1-\not{p}_3+m)\gamma^\mu(\not{p}_3+m)...$$
$$...\gamma_\nu(\not{p}_1-\not{p}_4+m)(\not{p}_4-m)] \quad (8.40)$$
$$= \frac{16e^4}{(t-m^2)(u-m^2)}\{3m^4+m^2(3s+2t-2u)-s(s+t+u)\} \quad (8.41)$$

Finally we have

$$\overline{|\mathcal{M}|^2} = 8e^4\left\{-\frac{13m^4+m^2(-5s+12t+4u)-tu}{(t-m^2)^2}\right.$$
$$-\frac{m^4+m^2(3s+4t+5u)-tu}{(u-m^2)^2}$$
$$\left.+\frac{3m^4+m^2(3s+2t-2u)-s(s+t+u)}{(t-m^2)(u-m^2)}\right\} \quad (8.42)$$

At ultra-relativistic limit, we have

$$\overline{|\mathcal{M}|^2} \xrightarrow{m\to 0} 8e^4\left\{-\frac{2s(s+t+u)}{st}+\frac{t}{s}+\frac{s}{t}\right\} \quad (8.43)$$

See FeynCalc code at *pair.nb*.

8.5 Furry theorem

The internal electron and position propagator give the same contribution to the amplitude, i.e., in case of t-channel Compton scattering

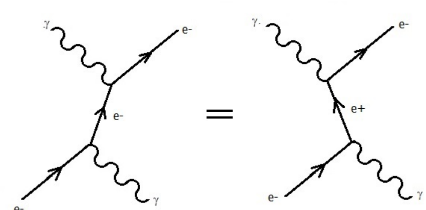

Figure 8.5: Furry theorem.

Exercise (8.1): Prove Furry theorem explicitly, using diagrams in figure (8.5).