9 One-Loop Corrections of Scalar ϕ^3-Interaction

9.1 One-loop diagrams

The major contribution of one-loop level connected diagrams of $\phi\phi \to \phi\phi$ interaction in scalar ϕ^3-interaction appear in figure (9.1).

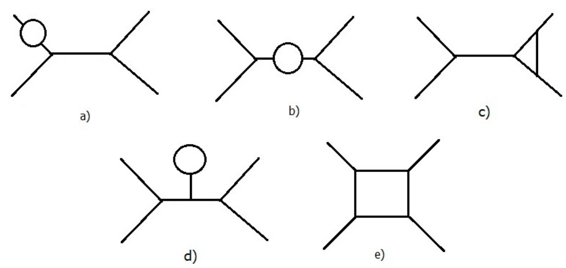


Figure 9.1: Major contribution of one-loop level diagrams.

There are field propagator correction in figure (9.1a,b), vertex correction in figure (.1c), tadpole diagram in figure (9.1d), and box diagram in figure (9.1e). The LSZ reduction expression of the amplitude of figure (9.1b) is

$$
\begin{aligned}
M_b^{(1)} &= (-i)^4 \frac{(-ig)^4}{(3!)^4 4!} \int d^4x_1 \int d^4x_2 \int d^4y_1 \int d^4y_2 \\
&\quad \times e^{-ip_1\cdot x_1 - ip_2\cdot x_2 + iq_1\cdot y_1 + iq_2\cdot y_2} \int d^4z_1 \int d^4z_2 \int d^4z_3 \int d^4z_4 \\
&\quad \times(\partial_{x_1}^2 + m^2)(\partial_{x_2}^2 + m^2)(\partial_{y_1}^2 + m^2)(\partial_{y_2}^2 + m^2) \\
&\quad \times\Delta(x_1, z_1)\Delta(x_2, z_1)\Delta(z_1, z_2)\Delta(z_2, z_3) \\
&\quad \times\Delta(z_2, z_3)\Delta(z_3, z_4)\Delta(z_4, y_1)\Delta(z_4, y_2) \qquad (9.1) \\
&= (-i)^4 \frac{(-ig)^4}{(3!)^4 4!} \int d^4z_1 \int d^4z_2 \int d^4z_3 \int d^4z_4 e^{-i(p_1+p_2)\cdot z_1 + i(q_1+q_2)\cdot z_4} \\
&\quad \times\Delta(z_1, z_2)\Delta(z_2, z_3)\Delta(z_2, z_3)\Delta(z_3, z_4) \\
&= (-i)^4 \frac{(-ig)^4}{(3!)^4 4!} \int d^4z_1 \int d^4z_2 \int d^4z_3 \int d^4z_4 e^{-i(p_1+p_2)\cdot z_1 + i(q_1+q_2)\cdot z_4} \\
&\quad \times \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int \frac{d^4k_3}{(2\pi)^4} \int \frac{d^4k_4}{(2\pi)^4} \Delta(k_1)\Delta(k_2)\Delta(k_3)\Delta(k_4) \\
&\quad \times e^{ik_1\cdot(z_1-z_2) + ik_2\cdot(z_2-z_3) + ik_3\cdot(z_2-z_3) + ik_4\cdot(z_3-z_4)} \qquad (9.2)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & (-i)^4 \frac{(-ig)^4}{(3!)^4 4!} \int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int \frac{d^4k_3}{(2\pi)^4} \int \frac{d^4k_4}{(2\pi)^4} \\
& \times \Delta(k_1)\Delta(k_2)\Delta(k_3)\Delta(k_4) \underbrace{\int d^4z_1 e^{-i(p_1+p_2-k_1)\cdot z_1}}_{=(2\pi)^4\delta^{(4)}(p_1+p_2-k_1)} \\
& \times \underbrace{\int d^4z_2 e^{-i(k_1-k_2-k_3)\cdot z_2}}_{=(2\pi)^4\delta^{(4)}(k_3-k_2-k_1)} \underbrace{\int d^4z_3 e^{-i(k_2-k_3-k_4)\cdot z_3}}_{=(2\pi)^4\delta^{(4)}(k_3-k_2+k_4)} \\
& \times \underbrace{\int d^4z_4 e^{-i(k_4-q_1-q_2)\cdot z_4}}_{=(2\pi)^4\delta^{(4)}(k_4-q_1-q_2)} \qquad (9.3) \\
= \quad & (-i)^4 \frac{(-ig)^4}{(3!)^4 4!} (2\pi)^4\delta^{(4)}(p_1+p_2-q_1-q_2) \\
& \times \int \frac{d^4k_2}{(2\pi)^4} \Delta(p_1+p_2)\Delta(k_2)\Delta(p_1+p_2-k_2)\Delta(p_1+p_2) \qquad (9.4)
\end{aligned}
$$

Note that k_2 is called the *loop momentum*, and what we have derived in (9.5) is an expression of one-loop diagram on momentum space with integration overall loop momentum. So that we have to add *additional Feynman rules* for loop diagram in momentum space with

- integrate overall loop momentum.

We may pay attention of our study on loop corrections of field propagator and interaction vertex, see figure (9.2), which will be developed to be *renormalization process* of field model parameters to be the physical parameters for measurement.

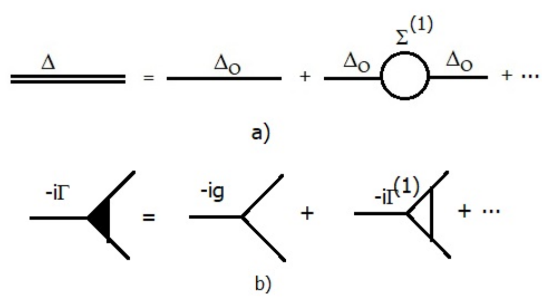


Figure 9.2: One-loop correction of field propagator and interaction vertex.

9.2 Field propagator correction and self-energy

The loop correction of field propagator is called *self-energy*, $-i\Sigma(k^2)$, in general we can write

$$\begin{aligned} \Delta = \Delta_0 + \Delta_0 + \Delta_0(-i\Sigma)\Delta_0 + \Delta_0(-i\Sigma)\Delta_0(-i\Sigma)\Delta_0 + ... \\ = \Delta_0 + \Delta_0(-i\Sigma)\left(\Delta_0 + \Delta_0(-i\Sigma)\Delta_0 + ...\right) \\ \mapsto \Delta = \Delta_0 + \Delta_0(-i\Sigma)\Delta \quad (9.5) \\ \Delta_0(\Delta_0^{-1} + i\Sigma)\Delta = \Delta_0 \rightarrow \Delta^{-1} = \Delta_0^{-1} + i\Sigma \quad (9.6) \\ \text{With } \Delta_0^{-1} = -i(p^2 - m^2 + i\epsilon) \mapsto \Delta^{-1} = -i(p^2 - m^2 - \Sigma(p^2) + i\epsilon) \quad (9.7) \end{aligned}$$

What we have derived is called *Dyson equation.* Within the perturbation theory, i.e. loop expansion, we can write

$$\Sigma(k^2) = \sum_{\#\ Loop\ n=1}^{\infty} \Sigma^{(n)}(k^2) \tag{9.8}$$

For example of one-loop self energy $\Sigma^{(1)}(k)$, its diagram on momentum space with truncated external leg field propagators appear in figure (9.3).

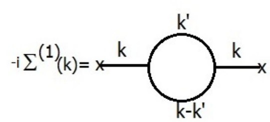


Figure 9.3: One-loop self-energy diagram.

Its expression is

$$-i\Sigma^{(1)}(k^2) = (-ig)^2 \int \frac{d^4k'}{(2\pi)^4} \frac{i}{k'^2 - m^2} \frac{i}{(k-k')^2 - m^2} \tag{9.9}$$

Apply with Feynman integral formula

$$\frac{1}{A_1 A_2 ... A_n} = (n-1)! \int_0^1 dx_1 ... dx_n \frac{\delta\left(1 - \sum_{i=1}^n x_i\right)}{(x_1 A_1 + x_2 a_2 + ... + x_n A_n)^n}$$

We will have from above

$$-i\Sigma^{(1)}(k^2) = -g^2 \int_0^1 dx \int \frac{d^4k'}{(2\pi)^4} \frac{1}{[x(k^2 - m^2) + (1-x)((k-k')^2 - m^2)]^2} \tag{9.10}$$

Expand the denominator term

$$x(k'^2 - m^2) + (1-x)((k-k')^2 - m^2)$$

$$= k'^2 - 2(1-x)kk' + (1-x)k^2 - m^2 = (k' - (1-x)k)^2 - m^2 + x(1-x)k^2$$

and shift loop momentum $k' \to k' + (1-x)k$, then we have

$$-i\Sigma^{(1)}(k^2) = -g^2 \int_0^1 dx \int \frac{d^4k'}{(2\pi)^4} \frac{1}{[k'^2 - M^2]^2}, \; M^2(x,p^2) = m^2 - x(1-x)k^2 \quad (9.11)$$

Next step w apply Wick's rotation of Minkowski momentum $k^\mu(\mu = 0,1,2,3)$ into Euclidean momentum $\tilde{k}^i(i = 1,2,3,4)$

$$k^0 \to i\tilde{k}^4 \mapsto d^4k' \to id^4\tilde{k} \text{ and } k'^2 = -\tilde{k}^2$$

Doing Euclidean momentum integration within spherical coordinate

$$d^4\tilde{k} = \tilde{k}^3 d\tilde{k} d\Omega_4 = \frac{1}{2}\tilde{k}^2 d\tilde{k}^2 d\Omega_4$$

From (9.11), we have

$$-i\Sigma^{(1)}(k^2) = -i\frac{g^2}{2}\Omega_4 \int_0^1 dx \int_0^\infty d\tilde{k}^2 \frac{\tilde{k}^2}{(\tilde{k}^2 + M^2)^2}$$
$$= -i\frac{g^2}{2}\Omega_4 \int_0^1 dx \left(\int_0^\infty \frac{d\tilde{k}^2}{\tilde{k}^2 + M^2} - M^2 \int_0^\infty \frac{d\tilde{k}^2}{(\tilde{k}^2 + M^2)^2} \right) \quad (9.12)$$

We have UV divergence of integration on the first term. To cure this we just apply the cutoff momentum Λ^2, with $\Lambda^2 \to \infty$, to the upper limit of the integration, then we have

$$\Sigma^{(1)}(k^2, \Lambda) = \frac{g^2}{2}\Omega_4 \int_0^1 dx \left(\ln\frac{\Lambda^2}{M^2} - 1 \right) = g^2\pi^2 \int_0^1 dx \left(\ln\frac{\Lambda^2}{M^2} - 1 \right) \quad (9.13)$$

Note that Ω_d is solid angle or surface area of unit sphere in d-dimensions and generally equal to

$$\Omega_d = \int_0^{2\pi} d\phi \int_0^\pi \sin\theta_1 d\theta_1 ... \int_0^\pi \sin^{d-1}\theta_{d-1} d\theta_{d-1} = \frac{2\pi^{d/2}}{\Gamma[d/2]}$$

9.3 Vertex correction

Within perturbation theory, i.e. loop expansion, we can write vertex correction as

$$\Gamma = \sum_{\#loop=0}^{\infty} \Gamma^{(n)}, \text{ with } \Gamma^{(0)} = 1 \quad (9.14)$$

Let us determine the one-loop vertex correction diagram, with labeled momentum and truncated external field propagator legs as appear in figure (9.4).

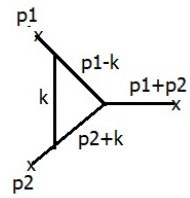


Figure 9.4: One-loop vertex correction diagram.

Its expression is

$$-ig\Gamma^{(1)}(p_1,p_2) = (-ig)^3 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2} \frac{i}{(p_1+k)^2-m^2} \frac{i}{(p_2+k)^2-m^2} \tag{9.15}$$

$$= 2g^3 \int_0^1 d\alpha \int_0^1 d\beta \int_0^1 d\gamma \int \frac{d^4k}{(2\pi)^4} \times \frac{\delta(1-\alpha-\beta-\gamma)}{[\alpha(k^2-m^2)+\beta((p_1-k)^2-m^2)+\gamma((p_2+k)^2-m^2)]^3} \tag{9.16}$$

Expand the denominator term

$$\begin{aligned} &\alpha(k^2-m^2)+\beta((p_1-k)^2-m^2)+\gamma((p_2+k)^2-m^2) \\ &= k^2 - 2k\cdot(p_1\alpha - p_2\beta) - m^2 \\ &= [k-(\alpha p_1 - \beta p_2)]^2 - [\underbrace{(\alpha p_1 - \beta p_2)^2 + m^2}_{=M^2(\alpha,\beta,p_1,p_2,m^2)}] \end{aligned}$$

and shift momentum $k \to k+(\alpha p_1 - \beta p_2)$, after integrate $d\gamma$ using delta function, we will have

$$\Gamma^{(1)} = 2ig^2 \int_0^1 d\alpha \int_0^1 d\beta \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2-M^2]^3} \tag{9.17}$$

Applying with Wick's rotation from Minkowski momentum into Euclidean momentum, and do the integration in spherical coordinates, then we have

$$\begin{aligned} \Gamma^{(1)} &= g^2\Omega_4 \int_0^1 d\alpha \int_0^1 d\beta \int_0^\infty d\tilde{k}^2 \frac{\tilde{k}^2}{[\tilde{k}^2+M^2]^3} \\ &= g^2\Omega_4 \int_0^1 d\alpha \int_0^1 d\beta \left\{ \int_0^\infty \frac{d\tilde{k}^2}{[\tilde{k}^2+M^2]^2} - M^2 \int_0^\infty \frac{d\tilde{k}}{[\tilde{k}^2+M^2]^3} \right\} \\ &= g^2\Omega_4 \int_0^1 d\alpha \int_0^1 d\beta \left\{ \frac{1}{M^2} - \frac{1}{2M^2} \right\} \\ &= \frac{g^2\Omega_4}{2} \underbrace{\int_0^1 d\alpha \int_0^1 d\beta \frac{1}{(\alpha p_1 - \beta p_2 + m^2)^2}}_{=I(p_1,p_2,m^2)} = \frac{g^2\Omega_4}{2} I(p_1,p_2,m^2) \end{aligned} \tag{9.18}$$

There is no divergence in one-loop vertex correction. The one-loop vertex will be written in the form

$$\mapsto -ig\Gamma \sim -ig\left(1 + g^2\frac{\Omega_4}{2}I(p_1, p_2, m^2)\right) \tag{9.19}$$

9.4 Methods of loop momentum integral regularization

Major problem of loop momentum integration is UV divergence. We cure this problem by regularize the integral with momentum cutoff at high energy. There are other loop momentum integral regularization methods always used in quantum field calculation.

- **Pauli-Villars regularization:** Let us determine the integral

$$I(p^2) = (-ig)^2\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}\frac{i}{(p-k)^2-m^2} \tag{9.20}$$

We known from above that this integral has UV divergence. To get rid it, we replace one of the field propagator in the form

$$\frac{i}{k^2-m^2} \to \frac{i}{k^2-m^2} - \frac{i}{k^2-\Lambda^2},\ \Lambda^2 \to \infty$$
$$= \frac{i(m^2-\Lambda^2)}{(k^2-m^2)(k^2-\Lambda^2)} \tag{9.21}$$

Now the integral $I(p^2)$ will become

$$\begin{aligned} I(p^2) \to I(p^2, M^2) &= i(-ig)^2(m^2-\Lambda^2) \\ &\times\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}\frac{i}{k^2-\Lambda^2}\frac{i}{(p-k)^2-m^2} \end{aligned} \tag{9.22}$$

It is divergence free as we have observe in (9.16-18). This method were introduced by W. Pauli and F. Villars in (1949).[1]

- **Dimensional regularization:** Since the divergence appears in $d = 4$ dimensions. We can we can get oblique it by doing the integration in dimension $d \neq 4$. Let us determine the same integral in (9.20) but now do the integration in dimension $d = 4-2\epsilon$, with $\epsilon \to 0$, and with dimensional correction factor μ for integration measure, as

$$I(p^2, d, \mu) = (-ig)^2\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2}\frac{i}{(p-k)^2-m^2} \tag{9.23}$$

$$= g^2\mu^{4-d}\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{1}{D^2} \tag{9.24}$$

[1] Pauli, W.; Villars, F. (1949). "On the Invariant Regularization in Relativistic Quantum Theory". Reviews of Modern Physics. 21 (3): 434–444.

when

$$
\begin{aligned}
D &= x[k^2 - m^2] + (1-x)[(p-k)^2 - m^2] \\
&= k^2 - 2(1-x)k\cdot p + (1-x)p^2 - m^2 \\
&= [k-(1-k)p]^2 - (1-x)^2p^2 + (1-x)p^2 - m^2 \\
&= [k-(1-x)p]^2 + x(1-x)p^2 - m^2 \qquad (9.25)
\end{aligned}
$$

$$
\text{Shift momentum} \xrightarrow[fixed\ p]{k^\mu \to k^\mu + (1-x)p^\mu} k^2 - \underbrace{(m^2 - x(1-x)p^2)}_{=M^2(x,m^2,p^2)} \qquad (9.26)
$$

Then we have from (9.24)

$$
\begin{aligned}
I(p^2, d, \mu) &= g^2\mu^{4-d}\int_0^1 dx \int \frac{d^dk}{(2\pi)^d}\frac{1}{[k^2 - M^2]^2} \\
&\xrightarrow{Wick's\ rotation} i\frac{g^2}{2}\mu^{4-d}\frac{\Omega_d}{(2\pi)^d}\int_0^1 dx\int_0^\infty d\tilde{k}^2\frac{(\tilde{k}^2)^{d/2-1}}{[\tilde{k}^2 + M^2]^2} \qquad (9.27)
\end{aligned}
$$

Insertion with $d = 4 - 2\epsilon$, when $\epsilon \to 0$, then (9.27) becomes

$$
I(p^2, \epsilon, \mu) = i\frac{g^2}{2}\mu^{2\epsilon}\frac{\Omega_{4-2\epsilon}}{(2\pi)^{4-2\epsilon}}\int_0^1 dx\int_0^\infty d\tilde{k}^2\frac{(\tilde{k}^2)^{1-\epsilon}}{[\tilde{k}^2 + M^2]^2} \qquad (9.28)
$$

Using the fact that

$$
\Omega_d = \frac{2\pi^{d/2}}{\Gamma[d/2]} \mapsto \Omega_{4-2\epsilon} = \frac{2\pi^{2-\epsilon}}{\Gamma[2-\epsilon]}
$$

$$
\int_0^\infty dt^2\frac{(t^2)^{d/2-1}}{[t^2 + a^2]^n} = \frac{2}{(a^2)^{n-d/2}}\frac{\Gamma[d/2]\Gamma[n-d/2]}{\Gamma[n]}
$$

We will have from above

$$
\begin{aligned}
I(p^2, \epsilon, \mu) &= \frac{ig^2}{16\pi^4}(4\pi^2\mu^2)^\epsilon\frac{2\pi^{2-\epsilon}}{\Gamma[2-\epsilon]}\frac{\Gamma[2-\epsilon]\Gamma[\epsilon]}{\Gamma[2]}\int_0^1 dx\frac{1}{(M^2)^\epsilon} \qquad (9.29) \\
&= \frac{ig^2}{8\pi^2}(4\pi\mu^2)^\epsilon\Gamma[\epsilon]\int_0^1 dx\frac{1}{M^{2\epsilon}(x, m^2, p^2)} \qquad (9.30)
\end{aligned}
$$

Using identity

$$
\Gamma[\epsilon] \sim \frac{1}{\epsilon} - \gamma_E + O(\epsilon),\ \gamma_E = 0.5772..\ \text{Euler's constant}
$$

$$
\frac{1}{(a^2)^\epsilon} \sim 1 - \epsilon\ln a^2 + O(\epsilon^2)
$$

Finally we have

$$I(p^2,\epsilon,\mu) = \frac{ig^2}{8\pi^2}\left(\frac{1}{\epsilon} - \gamma_E + (\epsilon)\right)\left(1+\epsilon\ln(4\pi\mu^2)+...\right)$$
$$\times\left(1-\epsilon\int_0^1 dx \ln M^2(x,m^2,p^2)+..\right)$$
$$= \frac{ig^2}{8\pi^2}\left(\underbrace{\frac{1}{\epsilon} - \gamma_E + \ln(4\pi\mu^2)}_{divergence\ part} - \underbrace{\int_0^1 dx\ ln[m^2 - x(1-x)p^2]}_{finite\ part} + O(\epsilon)\right) \quad (9.31)$$

We have extract divergence part of the loop momentum integral out of the finite part. This will be very useful for the next step for hiding this divergence part into some physical parameters of the field model, within the process called *renormalization*.

9.5 Some useful formula for dimensional regularization

9.5.1 Euclidean integration

Let us determine the integration

$$I_1(n,r) = \int d^nk\frac{1}{(k^2+a^2)^r}$$

Using the identity of Gamma function

$$\alpha^{-s}\Gamma[s] = \int_0^\infty dx x^{s-1}e^{-\alpha x}$$

$$\mapsto I_1(n,r) = \frac{1}{\Gamma[r]}\int d^nk\int_0^\infty dx x^{r-1}e^{x(k^2+a^2)}$$
$$= \frac{\pi^{n/2}}{\Gamma[r]}\int_0^\infty dx x^{r-1-n/2}e^{-xa^2}$$
$$= \pi^{n/2}a^{n-2r}\frac{\Gamma[r-n/2]}{\Gamma[r]}$$

Another integration is

$$I_2(n,r) = \int d^nk\frac{k^2}{(k^2+a^2)^r}$$

The trick is that first let us define

$$I_1(n,r,\alpha) = \int d^nk\frac{1}{(\alpha k^2+a^2)^r} = \alpha^{-n/2}I_1(n,r)$$

Differentiate w.r.t α and set $\alpha = 1$, we will have

$$I_2(n,r) = \frac{n\pi^{n/2}a^{n-2r+2}}{2(r-1)}\frac{\Gamma[r-1-n/2]}{\Gamma[r-1]}$$

9.5.2 Gamma function

By definition

$$\Gamma[x] = \int_0^\infty t^{x-1} e^{-t} dt \mapsto \Gamma[x+1] = x\Gamma[x]$$

For integer n, we have

$$\Gamma[1] = 1,\ \Gamma[n] = (n-1)! \text{ for } n \geq 1$$

$$\Gamma[1/2] = \sqrt{\pi},\ \Gamma[1/2+n] = \frac{(2n-1)!!}{2^n}\sqrt{\pi},\ n \geq 1$$

And Laurent series expansion of the gamma function[2]

$$\lim_{z\to 0} \Gamma[z] = \frac{1}{z} - \gamma_E + \frac{1}{2}\left(\gamma_E^2 + \frac{\pi^2}{6}\right) z^2 + O(z^3)$$

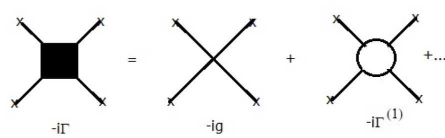
Exercises 9

9.1. Calculate the tadpole diagram $i\tau$, with truncated external legs, in figure (9.1d).

9.2. Calculate amplitude $i\mathcal{M}_d$ of box diagram in figure (9.1e).

9.3. Calculate one-loop correction of vertex function in ϕ^4-interaction model, with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{g}{4!}\phi^4$$



[2] https://math.stackexchange.com/questions/1287555/how-to-obtain-the-laurent-expansion-of-gamma-function-around-z-0