10 Scalar Field Renormalization

10.1 Introduction

Renormalization is a process of hiding divergence that occur in loop-order corrections into field model parameters, i.e. mass and coupling constant.

10.2 Minimal subtraction and counter term

Let us denote $\phi_0(x), m_0^2, g_0$ as the *bar quantities* of model field theory and $\phi(x), m^2, g$ are their corresponding *renormalized quantities*. The renormalization process start simply by introducing renormalization factor Z_ϕ of filed operator as

$$\phi_0(x) = \sqrt{Z_\phi}\phi(x) \tag{10.1}$$

Then the *bar field Lagrangian* will appear in the form

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 - \frac{1}{2}m_0^2\phi_0^2 - \frac{1}{3!}g_0\phi^3 \tag{10.2}$$

$$\xrightarrow{Z_\phi} \frac{Z_\phi^2}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m_0^2 Z_\phi\phi^2 - \frac{1}{3!}g_0 Z_\phi^{3/2}\phi^3 \tag{10.3}$$

Now let us assign the counter terms Lagrangian $\delta\mathcal{L}$ by writing

$$Z_\phi = 1 + \delta Z_\phi,\ m_0^2 Z_\phi = m^2 + \delta m^2,\ g_0 Z_\phi^{3/2} = g + \delta g \tag{10.4}$$

Then the bar Lagrangian can be written in terms of thr *renormalized Lagrangian* and counter term as

$$\begin{aligned} \mathcal{L}_0 &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{3!}g\phi^3 \\ &\quad + \frac{\delta Z_\phi}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\delta m^2\phi^2 - \frac{1}{3!}\delta g\phi^2 && (10.5) \\ &= \mathcal{L} + \delta\mathcal{L} && (10.6) \end{aligned}$$

In the renormalization process, all divergences will be factorized into the counter term. Under subtraction the renormalized Lagrangian will be free from them. This process is also known in the name of *minimal subtraction* renormaliztion.

We can assign Feynman rules for field propagator and vertex counter terms, see figure (10.1), as

- field propagator counter term $\delta\Delta(p) = i(\delta Z_\phi p^2 - \delta m^2)$
- interaction vertex counterm $-i\delta g$

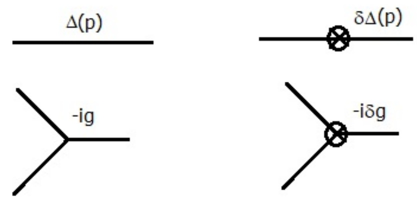

Figure 10.1: Field propagator and interaction vertex counter term diagrams.

10.3 Renormalization mass

Let us determine the interacting field propagator

$$\Delta_0(x,y) = \langle 0|T[\phi_0(x)\phi_0(y)]|0\rangle = Z_\phi \langle 0|[T\phi(x)\phi(y)]|0\rangle = Z_\phi \Delta(x,y) \quad (10.7)$$

Taylor expansion of self-energy about the renormalized mass squared m^2 as

$$\Sigma(p^2) = \Sigma(m^2) + (p^2 - m^2)\Sigma'(m^2) + \tilde{\Sigma}(p^2),\ \Sigma' = \frac{d\Sigma(p^2)}{dp^2} \quad (10.8)$$

where $\tilde{\Sigma}(p^2)$ is higher derivative terms of the expansion and satisfy $\tilde{\Sigma}(m^2) = 0$. The loop correction of bar field propagator becomes

$$\Delta_0(p^2) = \frac{i}{p^2 - m_0^2 - \Sigma(m^2) - (p^2 - m^2)\Sigma'(m^2) - \tilde{\Sigma}(p^2) + i\epsilon} \quad (10.9)$$

Let us define the renormalized mass squared as

$$m^2 = m_0^2 + \Sigma(m^2) \mapsto \delta m^2 = -\Sigma(m^2) \quad (10.10)$$

and let us rewrite

$$\tilde{\Sigma}(p^2) = (1 - \Sigma'(m^2))\tilde{\Sigma}(p^2)$$

Then we have from (10.9)

$$\begin{aligned} \Delta_0(p) &= \frac{i}{(p^2 - m^2 + i\epsilon)(1 - \Sigma'(m^2) - \tilde{\Sigma}(m^2))} \\ &\simeq \frac{i}{p^2 - m^2 - \tilde{\Sigma}(p^2) + i\epsilon}\left(1 - \Sigma'(m^2)\right)^{-1} \equiv Z_\phi \Delta(p) \quad (10.11) \\ \mapsto Z_\phi &= 1 + \delta Z_\phi = (1 - \Sigma'(m^2))^{-1} = 1 + \Sigma'(m^2) + ... \\ \delta Z_\phi &\simeq \Sigma'(m^2) \quad (10.12) \end{aligned}$$

In case of one-loop self-energy in ϕ^3-interaction within dimensional regularization of loop momentum integration, we have

$$
\begin{aligned}
-i\Sigma(p^2) &= (-ig)^2\mu^{4-d}\int \frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2}\frac{i}{(p-k)^2-m^2} \\
&= \frac{g^2\mu^{4-d}}{(2\pi)^d}\int_0^1 dx\int d^dk\frac{1}{[x(k^2-m^2)+(1-x)((p-k)^2-m^2)]^2} \quad (10.13)\\
&\xrightarrow{k\to k+(1-x)p} \frac{g^2\mu^{4-d}}{(2\pi)^d}\int_0^1 dx\int d^dk\frac{1}{k^2-M^2},\ M^2=m^2-x(1-x)p^2 \quad (10.14)\\
&\xrightarrow{k^0\to i\tilde{k}^4} \frac{ig^2\mu^{4-d}}{2(2\pi)^d}\Omega_d\int_0^1 dx\int_0^\infty d\tilde{k}^2\frac{\tilde{k}^2}{[\tilde{k}^2+M^2]^2} \quad (10.15)\\
&= \frac{ig^2}{2}\frac{\mu^{4-d}}{(2\pi)^d}\frac{2\pi^{d/2}}{\Gamma[d/2]}\frac{\Gamma[d/2]\Gamma[2-d/2]}{\Gamma[2]}\int_0^1 dx\frac{1}{(M^2)^{d/2-2}} \quad (10.16)\\
&\xrightarrow{d=4-2\epsilon} \frac{ig^2}{16\pi^2}(4\pi\mu^2)^\epsilon\Gamma[\epsilon]\int_0^1 dx\frac{1}{M^{2\epsilon}} \quad (10.17)\\
&\simeq \frac{ig^2}{16\pi^2}\left(1+\epsilon\ln(4\pi\mu^2)+...\right)\left(\frac{1}{\epsilon}-\gamma_E+O(\epsilon)\right)\\
&\quad\times\int_0^1 dx\left(1-\epsilon\ln M^2+...\right) \quad (10.18)
\end{aligned}
$$

Finally we have

$$
-i\Sigma(p^2)\simeq\frac{ig^2}{16\pi^2}\left(\frac{1}{\epsilon}-\gamma_E+\ln 4\pi\mu^2+..\right)-\int_0^1 dx\ln M^2(x,p^2)+... \quad (10.19)
$$

10.4 Callan-Symanzik equation

We start from n-point correlation function

$$
\begin{aligned}
G^{(n)}(x_1,...,x_n) &= \langle 0|T[\phi(x_1)...\phi(x_n)]|0\rangle \\
&= Z_\phi^{-n/2}\langle 0|T[\phi_0(x_1)...\phi_0(x_n)]|0\rangle \quad (10.20)\\
&= Z_\phi^{-n/2}G_0^{(n)}(x_1,...,x_n) \quad (10.21)
\end{aligned}
$$

Since tghe bare correlation function $G_0^{(n)}=G_0^{(n)}(\phi_0,g_0,m_0)$ and the renormlizaed correlation function $G^{(n)}=G^{(n)}(\phi,m,g,\mu)$. Then we will have

$$
\frac{dG_0^{(n)}}{d\mu}=0 \quad (10.22)
$$

Actually $\phi=\phi(\mu)$, $m=m(\mu)$ and $g=g(\mu)$, when we make the variation $\mu\to\mu+\delta\mu$, we will have the following variations

$$
\begin{aligned}
\phi\to\phi+\delta\phi &= (1+\delta\eta)\phi,\ \delta\eta=\delta\phi/phi \quad (10.23)\\
g &\to g+\delta g \quad (10.24)
\end{aligned}
$$

From above we will have

$$0 = \frac{d}{d\mu}\left(Z_\phi^{n/2} G^{(n)}\right) = \frac{\partial G^{(n)}}{d\mu} + \frac{\partial G^{(n)}}{\partial g}\frac{dg}{d\mu} - n\frac{d\eta}{d\mu}G^{(n)} \tag{10.25}$$

after we have used the fact that $Z_\phi^{1/2} = 1 - \delta\eta$. We can write (10.25) as

$$\left(\mu\frac{\partial}{\partial\mu} + \mu\frac{\partial g}{\partial\mu}\frac{\partial}{\partial g} - n\mu\frac{\partial\eta}{\partial\mu}\right) G^{(n)}(x_1, ..., x_n; g, \mu) = 0 \tag{10.26}$$

Next we define

$$\beta = \mu\frac{\partial g}{\partial\mu}, \;\; \gamma = -\mu\frac{\partial\eta}{\partial\mu} \tag{10.27}$$

Then (10.26) becomes

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial g} + n\gamma\right) G^{(n)}(x_1, ..., x_n; g, \mu) = 0 \tag{10.28}$$

This equation is known in the name of *Callan-Symanzik equation.* Note that the renormalized correlation function is directly depends of the renormalization scale μ, the β function measure how the coupling constant depends of the renormalization scale and the γ function measure the μ dependent of the field operator renormalization.

10.5 Beta function

By definition in (10.27)

$$\beta(g) = \mu\frac{\partial g}{\partial\mu} \mapsto \int_{g_0}^{g}\frac{dg}{\beta(g)} = \ln\frac{\mu}{\mu_0} \tag{10.29}$$

with $g(\mu_0) = g_0$. Generic behavior of $\beta(g)$ appears in figure (10.1), and g_1, g_2 are called *fixed points.*

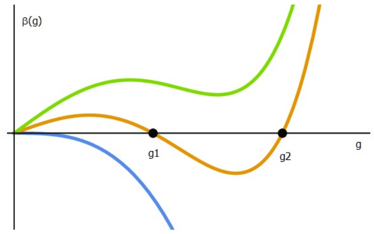

Figure 10.2: Behavior of the $\beta(g)$.

a) In case of real massive scalar ϕ^4-interaction we will have

$$\beta(g) = \frac{3g^2}{16\pi^2} + O(g^3) \tag{10.30}$$

We will have from (10.29)

$$g = \frac{g_0}{1 - \frac{3}{16\pi^2} g_0 \ln \frac{\mu}{\mu_0}} \tag{10.31}$$

b) In case of the $\beta(g)$ encounter fixed points g_1, g_2, i.e., near the fixed point g_1 we may assume

$$\beta(g) = a(g_1 - g), \quad g_0 \ < g < g_1 \tag{10.32}$$

$$(10.29) \mapsto g_1 - g \simeq \mu^{-a} \tag{10.33}$$

This shows that $g \xrightarrow{\mu\to\infty} g_1$.

c) In case of $\beta(g)$ is negative

$$\beta(g) = -ag^n, \quad a > 0,\ n > 1 \tag{10.34}$$

$$(10.29) \mapsto g = \frac{g_0}{[1 + g_0^{n-1}(n-1)a \ln \frac{\mu}{\mu_0}]^{1/(n-1)}} \tag{10.35}$$

This shows that $g \xrightarrow{\mu\to\infty} 0$. This is known as *asymptotic freedom* behavior, as appear in QCD.