11 One-Loop Corrections in QED

The loop mimentum calculation will be done with dimensional regularization in dimension $d = 4 - 2\epsilon$.

11.1 Electron self-energy

11.2 Vacuum polarization

One-loop photon self-energy or vacuum polarization diagram

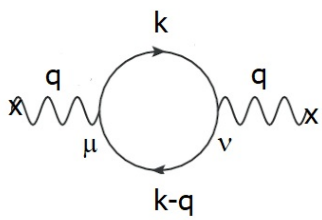

Figure 11.1: One-loop vacuum polarization.

Its expression is

$$\begin{aligned} -i\Pi^{\mu\nu}(q^2) &= (-)(-ie)^2\mu^{4-d}\int \frac{d^dk}{(2\pi)^d} Tr\left[\gamma^\mu \frac{i(\not{k}+m)}{k^2-m^2}\gamma^\nu \frac{i(\not{k}-\not{q}-m)}{(k-q)^2-m^2}\right] \\ &= -e^2\frac{\mu^{4-d}}{(2\pi)^d}\int d^dk \frac{Tr[\gamma^\mu(\not{k}+m)\gamma^\nu(\not{k}-\not{q}-m)]}{(k^2-m^2)((k-q)^2-m^2)} \end{aligned} \tag{11.1}$$

Apply with Feynman parametrized integral, we will have

$$\Pi^{\mu\nu}(q^2) = -ie^2\frac{\mu^{4-d}}{(2\pi)^d}\int_0^1 dx \int d^dk \frac{Tr[\gamma^\mu(\not{k}+m)\gamma^\nu(\not{k}-\not{q}-m)]}{[x(k^2-m^2)+(1-x)((k-q)^2-m^2)]^2} \tag{11.2}$$

Let us first evaluate the numerator of the integrand

$$N^{\mu\nu} = Tr[\gamma^\mu(\not{k}+m)\gamma^\nu(\not{k}-\not{q}-m)] = -4\{g^{\mu\nu}(k^2 - k\cdot q + m^2) + k^\mu(q^\nu - 2k^\nu + k^\nu q^\mu\}$$

Then let us evaluate the denominator of the integrand

$$\begin{aligned} D = x(k^2-m^2) + (1-x)((k-q)^2-m^2) &= k^2 - m^2 - 2(1-x)q\cdot k + (1-x)q^2 \\ &= (k-(1-x)q)^2 - \underbrace{[m^2 - (1-x)xq^2]}_{=A^2(x,m^2,q^2)} \end{aligned}$$

Now let us shift the loop momentum $k \to k + (1-x)q$, keeps q constant, then we have from above

$$N'^{\mu\nu} = -4\{g^{\mu\nu}[k^2 - (1-x)xq^2 + m^2] - 2k^\mu k^\nu - 2(1-x)q^\mu q^\nu\}$$

$$D = k^2 - A^2(x, m^2, q^2)$$

After we have eliminated all linear-k terms of the numerator. From (11.2), we can write

$$\Pi^{\mu\nu}(q^2) = g^{\mu\nu}\Pi_1(q^2) - \Pi_2^{\mu\nu}(q^2) \tag{11.3}$$

where

$$\begin{aligned}
\Pi_1(q^2) &= 4ie^2\frac{\mu^{4-d}}{(2\pi)^d}\int_0^1 dx \int d^d k \frac{k^2-(1-x)xq^2+m^2}{[k^2-A^2]^2} \\
&= 4ie^2\frac{\mu^{4-d}}{(2\pi)^d}\int_0^1 dx \int d^d k \frac{k^2}{[k^2-A^2]^2} \\
-4ie^2\frac{\mu^{4-d}}{(2\pi)^d}&\int_0^1 dx[(1-x)xq^2-m^2]\int d^d k\frac{1}{[k^2-A^2]^2}
\end{aligned} \tag{11.4}$$

$$\begin{aligned}
\xrightarrow[spherical\ coordinates]{Wick's\ rotation} & 4e^2\frac{\mu^{4-d}}{(2\pi)^d}\frac{\Omega_d}{2}\int_0^1 dx\int_0^\infty d\bar{k}^2\frac{(\bar{k}^2)^{d/2-1}\bar{k}^2}{[\bar{k}^2+A^2]^2} \\
+4e^2\frac{\mu^{4-d}}{(2\pi)^d}\frac{\Omega_d}{2}&\int_0^1 dx[(1-x)xq^2-m^2]\int_0^\infty d\bar{k}^2\frac{(\bar{k}^2)^{d/2-1}}{[\bar{k}^2+A^2]^2}
\end{aligned} \tag{11.5}$$

$$\begin{aligned}
\mapsto \Pi_1(q^2) &= 4e^2\frac{\mu^{4-d}}{(4\pi)^{d/2}}\frac{\Gamma[1-d/2]\Gamma[d/2+1]}{\Gamma[d/2]\Gamma[2]}\int_0^1 dx (A^2)^{d/2-1} \\
+4e^2\frac{\mu^{4-d}}{(4\pi)^{d/2}}&\frac{\Gamma[n-d/2]}{\Gamma[n]}\int_0^1 dx[(1-x)xq^2-m^2](A^2)^{d/2-2}
\end{aligned} \tag{11.6}$$

And

$$\begin{aligned}
\Pi_2^{\mu\nu}(q^2) &= 8ie^2\frac{\mu^{4-d}}{(2\pi)^d}\int_0^1 dx\int d^d k\frac{k^\mu k^\nu+(1-x)q^\mu q^\nu}{[k^2-A^2]^2} \\
&= 8ie^2\frac{\mu^{4-d}}{(2\pi)^d}\int_0^1\int d^d k\frac{k^\mu k^\nu}{[k^2-A^2]^2} \\
+8ie^2\frac{\mu^{4-d}}{(2\pi)^d}&\int_0^1 dx(1-x)q^\mu q^\nu\int d^d k\frac{1}{[k^2-A^2]^2}
\end{aligned} \tag{11.7}$$

$$\begin{aligned}
\xrightarrow[spherical\ coordinates]{Wick's\ rotation} & 8e^2\frac{\mu^{4-d}}{(2\pi)^d}\frac{\Omega_d}{2}\int_0^1 dx\int_0^\infty d\bar{k}^2\frac{(\bar{k}^2)^{d/2-1}\bar{k}^\mu\bar{k}^\nu}{[\bar{k}^2+A^2]^2} \\
-8e^2\frac{\mu^{4-d}}{(2\pi)^d}\frac{\Omega_d}{2}&\int_0^1 dx(1-x)q^\mu q^\nu\int_0^\infty d\bar{k}^2\frac{(\bar{k}^2)^{d/2-1}}{[\bar{k}^2+A^2]^2}
\end{aligned} \tag{11.8}$$

$$\begin{aligned}
\mapsto \Pi_2^{\mu\nu}(q^2) &= 4e^2\frac{\mu^{4-d}}{(4\pi)^d/2}\frac{\Gamma[1-d/2]\Gamma[d/2+1]}{\Gamma[d/2]\Gamma[2]}g^{\mu\nu}\int_0^1 dx(A^2)^{d/2-1} \\
-8e^2\frac{\mu^{4-d}}{(4\pi)^d/2}&\frac{\Gamma[2-d/2]}{\Gamma[2]}q^\mu q^\nu\int_0^1 dx(1-x)(A^2)^{d/2-2}
\end{aligned} \tag{11.9}$$

After we have used the integral formulas

$$\int_0^\infty d\bar{k}^2\frac{(\bar{k}^2)^{d/2-1}}{[\bar{k}^2+A^2]^n} = \frac{\Gamma[d/2]\Gamma[n-d/2]}{\Gamma[n]}(A^2)^{d/2-n}$$

$$\int_0^\infty d\bar{k}^2 \frac{(\bar{k}^2)^{d/2-1}(\bar{k}^2)^s}{[\bar{k}^2+A^2]^n} = \frac{\Gamma[n-d/2-s]\Gamma[d/2+s]}{\Gamma[n]}(A^2)^{d/2+s-n}$$

$$\int_0^\infty d\bar{k}^2 \frac{(\bar{k}^2)^{d/2-1}\bar{k}^\mu\bar{k}^\nu}{[\bar{k}^2+A^2]^n} = \frac{g^{\mu\nu}}{2}\frac{\Gamma[d/2]\Gamma[n-d/2-1]}{\Gamma[n]}(A^2)^{d/2+1-n}$$

In dimension $d = 4 - 2\epsilon$, we will have from (11.6)

$$\Pi_1(q^2) = \frac{e^2}{4\pi^2}(4\pi\mu^2)^\epsilon\Gamma[\epsilon-1](2-\epsilon)\int_0^1 dx A^2(A^2)^{-\epsilon}$$
$$+\frac{e^2}{4\pi^2}(4\pi\mu^2)^\epsilon\Gamma[\epsilon]\int_0^1 dx[(1-x)xq^2-m^2](A^2)^{-\epsilon} \tag{11.10}$$

After we have used the identity of Gamma function $\Gamma[z+1] = z\Gamma[z]$. And from (11.9)

$$\Pi_2^{\mu\nu}(q^2) = \frac{e^2}{4\pi^2}(4\pi\mu^2)^\epsilon\Gamma[\epsilon-1](2-\epsilon)g^{\mu\nu}\int_0^1 dx A^2(A^2)^{-\epsilon}$$
$$-\frac{e^2}{2\pi^2}(4\pi\mu^2)^\epsilon\Gamma[\epsilon]q^\mu q^\nu\int_0^1 dx(1-x)(A^2)^{-\epsilon} \tag{11.11}$$

From (11.3), we then have

$$\Pi^{\mu\nu}(q^2) = \frac{e^2}{4\pi^2}(4\pi\mu^2)^\epsilon\Gamma[\epsilon]\int_0^1 dx\{g^{\mu\nu}[(1-x)xq^2-m^2]+2(1-x)q^\mu q^\nu\}(A^2)^{-\epsilon} \tag{11.12}$$

Using identity

$$X^\epsilon = 1 + \epsilon\ln X + O(\epsilon)$$
$$\Gamma[\epsilon] = \frac{1}{\epsilon} - \gamma_E + O(\epsilon)$$

Then we have from above

$$\Pi^{\mu\nu}(q^2) = \frac{e^2}{4\pi^2}\left(\frac{1}{\epsilon} - \gamma_E + \ln\frac{4\pi\mu^2}{m^2}\right)$$
$$-\frac{e^2}{4\pi^2}\int_0^1 dx\{g^{\mu\nu}[(1-x)xq^2-m^2]+2(1-x)q^\mu q^\nu\}\ln[1-(1-x)xq^2/m^2]^2 \tag{11.13}$$

11.3 Vertex correction

One-loop correction of QED vertex appears in the following figure.

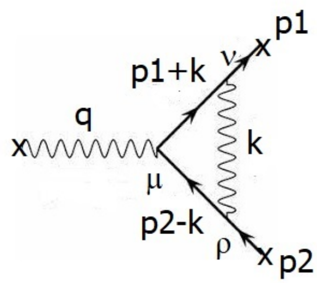

Figure 11.2: One-loop QED vertex.

Its expression is

$$-ie\Gamma^\mu(p_1,p_2) = (-ie)^3\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\gamma^\nu\frac{i(\not{p}_1+\not{k}+m)}{(p_1+k)^2-m^2}\gamma^\mu\frac{i(\not{p}_2-\not{k}-m)}{(p_2-k)^2-m^2}\gamma^\rho\frac{-ig_{\nu\rho}}{k^2} \quad (11.14)$$

$$= -e^3\frac{\mu^{4-d}}{(2\pi)^d}\int d^dk\frac{\gamma^\nu(\not{p}_1+\not{k}+m)\gamma^\mu(\not{p}_1-\not{k}-m)\gamma_\nu}{((p_1+k)^2-m^2)((p_2-k)^2-m^2)k^2} \quad (11.15)$$

Apply with Feynman parametrized integral, we have

$$\Gamma^\mu(p_1,p_2) = -2ie^2\frac{\mu^{4-d}}{(2\pi)^d}\int_0^1 dx\int_0^1 dy\int_0^1 dz\delta(1-x-y-z)$$
$$\times\int d^dk\frac{\gamma^\nu(\not{p}_1+\not{k}+m)\gamma^\mu(\not{p}_2-\not{k}-m)\gamma_\nu}{[x((p_1)+k)^2-m^2)+y((p_2)-k)^2-m^2)+zk^2]^3} \quad (11.16)$$

Evaluate the numerator of the integrand

$$N^\mu = \gamma^\nu(\not{p}_1+\not{k}+m)\gamma^\mu(\not{p}_2-\not{k}-m)\gamma_\nu = -2\not{p}_2\gamma^\mu\not{p}_1-(d-4)\not{p}_1\gamma^\mu\not{p}_2$$
$$-2\not{p}_2\gamma^\mu\not{k}-(d-4)\not{k}\gamma^\mu\not{p}_2+\not{k}\gamma^\mu\not{p}_1+(d-4)\not{p}_1\gamma^\mu\not{k}$$
$$+(d-2)\not{k}\gamma^\mu\not{k}-m^2(2-d)\gamma^\mu$$
$$-m\{(d-4)[\not{p}_1+\not{k}]\gamma^\mu+4[p_1^\mu+k^\mu]\}+m\{(d-4)\gamma^\mu[\not{p}_2-\not{k}]+4[p_2^\mu-k^\mu]\} \quad (11.17)$$

Evaluate the denominator of the integrand

$$D = x((p_1+k)^2-m^2)+y((p_2-k)^2-m^2)+zk^2 = k^2+xp_1-yp_2-(x+y)m^2+xp_1^2+yp_2^2$$

$$= (k+(xp_1-yp_2))^2-(x+y)m^2+(1-x)xp_1^2+(1-y)yp_2^2$$

Shift loop momentum $k\to k+(xp_1-yp_2)$, then we have

$$N'^\mu = -2\not{p}_2\gamma^\mu\not{p}_1-(d-4)\not{p}_1\gamma^\mu\not{p}_2-m(d-4)[\not{p}_1\gamma^\mu-\gamma^\mu\not{p}_2]$$

$$-4m[p_1^\mu-p_2^\mu]-m^2(2-d)\gamma^\mu+(d-2)\not{k}\gamma^\mu\not{k}$$

$$D = k^2 - \underbrace{[(x+y)m^2 - (1-x)xp_1^2 - (1-y)yp_2^2]}_{=A^2} \equiv k^2 - A^2(x,y,m^2,p_1^2,p_2^2)$$

where we have eliminated all linear k-terms in N'^{μ}. From (11.6), after integrate dz using delta function, we will have

$$\Gamma^{\mu}(p_1,p_2) = -2ie^2\frac{\mu^{4-d}}{(2\pi)^d}\int_0^1 dx\int_0^1 dy\int d^dk\frac{N'^{\mu}}{[k^2-A^2]^3} \tag{11.18}$$

$$\equiv \Gamma_0^{\mu}(p_1,p_2) + \Gamma_1^{\mu\nu}(p_1,p_2) \tag{11.19}$$

when

$$\Gamma_0^{\mu}(p_1,p_2) = -2ie^2\{-2\not{p}_2\gamma^{\mu}\not{p}_1 - (d-4)\not{p}_1\gamma^{\mu}\not{p}_2 - m(d-4)[\not{p}_1\gamma^{\mu} - \gamma^{\mu}\not{p}_2] -4m[p_1^{\mu}-p_2^{\mu}]\}\frac{\mu^{4-d}}{(2\pi)^d}\int_0^1 dx\int_0^1 dy\int d^dk\frac{1}{[k^2-A^2]^3} \tag{11.20}$$

$$\Gamma_1^{\mu}(p_1,p_2) = -2ie^2(d-2)\frac{\mu^{4-d}}{(2\pi)^d}\int_0^1 dx\int_0^1 dy\int d^dk\frac{\not{k}\gamma^{\mu}\not{k}}{[k^2-A^2]^3} \tag{11.21}$$

$$= -2ie^2(d-2)\frac{\mu^{4-d}}{(2\pi)^d}\int_0^1 dx\int_0^1 dy\int d^dk\frac{2\gamma_{\nu}k^{\nu}k^{\mu} - \gamma_{\nu}\gamma_{\rho}\gamma^{\mu}k^{\nu}k^{\rho}}{[k^2-A^2]^3} \tag{11.22}$$

Note that Γ_0^{μ} is finite under loop momentum integration in 4-dimensions, so that we need not to do integral regularization. Anyway Γ_1^{μ} still have logarithmic UV divergence, and integral regularization is still required.

At $d=4$ we will have, after applying with Wick's rotation,

$$\Gamma_0^{\mu}(p_1,p_2) = \frac{e^2}{4\pi^2}\{\not{p}_2\gamma^{\mu}\not{p}_1 + 2m[p_1^{\mu}-p_2^{\mu}]\}\int_0^1 dx\int_0^1 dy\int_0^{\infty} d\bar{k}^2\frac{\bar{k}^2}{[\bar{k}^2+A^2]^3}$$

$$= \frac{e^2}{8\pi^2}\{\not{p}_2\gamma^{\mu}\not{p}_1 + 2m[p_1^{\mu}-p_2^{\mu}]\}\int_0^1 dx\int_0^1 dy\frac{1}{A^2(x,y,m^2,p_1^2,p_2^2)} \tag{11.23}$$

At $d = 4-2\epsilon$ we will have, after applying with Wick's rotation,

$$\Gamma_1^{\mu}(p_1,p_2) = -\frac{e^2}{\pi^2}(1-\epsilon)\frac{(4\pi\mu^2)^{\epsilon}}{\Gamma[d/2]}\int_0^1 dx\int_0^1 dy\int_0^{\infty} d\bar{k}^2\frac{2\gamma_{\nu}\bar{k}^{\nu}\bar{k}^{\mu} - \gamma_{\nu}\gamma_{\rho}\gamma^{\mu}k^{\nu}\bar{k}^{\rho}}{[\bar{k}^2+A^2]^3}$$

$$= \frac{e^2}{2\pi^2}(1-\epsilon)^2(4\pi\mu^2)^{\epsilon}\Gamma[\epsilon]\gamma^{\mu}\int_0^1 dx\int_0^1 dy(A^2)^{-\epsilon} \tag{11.24}$$

$$\simeq \frac{e^2}{2\pi^2}\gamma^{\mu}\left(\frac{1}{\epsilon} - \gamma_E + \ln\frac{4\pi\mu^2}{m^2}\right) -\frac{e^2}{2\pi^2}\gamma^{\mu}\int_0^1 dx\int_0^1 dy\ln\left[(x+y) - (1-x)x\frac{p_1^2}{m^2} - (1-y)y\frac{p_2^2}{m^2}\right] + O(\epsilon) \tag{11.25}$$

The first part diverges at $\epsilon \to 0$, and the second part is finite.