11 One-Loop Corrections of QED

QED is a quantum theory of electron-photon interaction. The 1-loop corrections, within perturbation theory, consist of

- electron-self energy
- photon-self energy or vacuum polarization
- vertex correction

See figure (11.1) below. The loop momentum calculation will be done with dimensional regularization.

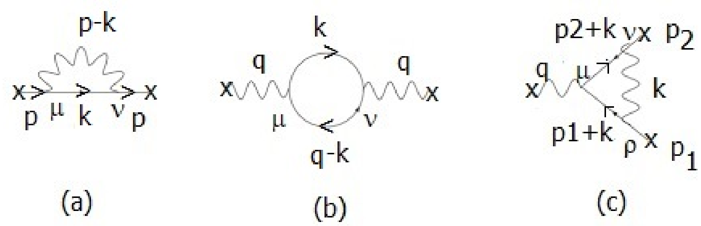

Figure 11.1: 1-loop correction in QED.

11.1 Electron self-energy

From figure (11.1a), the expression of electron self-energy is

$$-i\Sigma(p^2) = (-ie)^2\mu^{4-d}\int \frac{d^dk}{(2\pi)^d}\frac{-ig_{\mu\nu}}{(p-k)^2}\gamma^\mu \frac{i(\not{k}+m)}{k^2-m^2}\gamma^\nu \tag{11.1}$$

$$= e^2\frac{\mu^{4-d}}{(2\pi)^d}\int d^dk \frac{\gamma^\mu(\not{k}+m)\gamma_\mu}{(p-k)^2(k^2-m^2)} \tag{11.2}$$

$$= e^2\frac{\mu^{4-d}}{(2\pi)^d}\int_0^1 dx\int d^dk \frac{\gamma^\mu(\not{k}+m)\gamma_\mu}{[x(p-k)^2+(1-x)(k^2-m^2)]^2} \tag{11.3}$$

The denominator

$$x(p-k)^2+(1-x)(k^2-m^2 = k^2-2xp\cdot k+xp^2-(1-x)m^2$$

$$= (k-xp)^2-(1-x)[m^2-xp^2] \tag{11.4}$$

$$\xrightarrow{k\to k+xp} = k^2 - \underbrace{(1-x)[m^2-xp^2]}_{A(x,m^2,p^2)} \tag{11.5}$$

We will have from (11.3)

$$\Sigma(p^2) = ie^2\frac{\mu^{4-d}}{(2\pi)^d}\int_0^1 dx\int d^dk \frac{\gamma^\mu(\not{k}+x\not{p}+m)\gamma_\mu}{[k^2-A^2]^2} \tag{11.6}$$

Note that the survival terms in loop momentum integration in (11.6), with even-k denominator, must be even-k numerator. Then we have

$$\Sigma(p^2) = ie^2 \int_0^1 dx \left(\gamma^\mu \gamma^\rho \gamma_\mu p_\rho x + \gamma^\mu \gamma_\mu m\right) \frac{\mu^{4-d}}{(2\pi)^d} \int d^d k \frac{1}{[k^2 - A^2]^2} \tag{11.7}$$

In d-dimension, we will have

$$\gamma^\mu \gamma_\mu = d,\ \gamma^\mu \gamma^\rho \gamma_\mu = (2-d)\gamma^\rho$$

From above

$$\Sigma(p^2) = ie^2 \int_0^1 dx \left((2-d)x\not{p} + dm\right) \frac{\mu^{4-d}}{(2\pi)^d} \int d^d k \frac{1}{[k^2 - A^2]^2} \tag{11.8}$$

Apply Wick's rotation of the lopp momentum, $k^0 \to i\tilde{k}^4$, and do the Euclidean loop momentum integration in dimension $d = 4 - 2\epsilon$ using spherical coordinate, we will have from above

$$\begin{aligned} \Sigma(p^2) = -e^2 \int_0^1 dx[-(2-\epsilon)x\not{p} + (4-2\epsilon)m] \\ \times \frac{\mu^{2\epsilon}}{(2\pi)^{4-2\epsilon}} \frac{\Omega_{4-2\epsilon}}{2} \int_0^\infty d\tilde{k}^2 \frac{(\tilde{k}^2)^{d/2-1}}{[\tilde{k}^2 + A^2]^2} \end{aligned} \tag{11.9}$$

Using Dirac equation, and scalar integral

$$(\not{p} + m)U(p) = 0 \mapsto \not{p}U(p) = -mU(p)$$

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma[d/2]} \mapsto \Omega_{4-2\epsilon} = \frac{2\pi^{2-\epsilon}}{\Gamma[2-\epsilon]}$$

$$\int_0^\infty d\tilde{k}^2 \frac{(\tilde{k}^2)^{d/2-1}}{[\tilde{k}^2 + A^2]^n} = A^{d/2-n} \frac{\Gamma[d/2]\Gamma[n-d/2]}{\Gamma[n]}$$

Then we have from (11.9)

$$\Sigma(p^2) = -e^2 \left(\frac{4\pi\mu^2}{m^2}\right)^\epsilon \Gamma[\epsilon] \int_0^1 dx \frac{(2-\epsilon)(2+x)m}{[(1-x)(1-xp^2/m^2)]^\epsilon} \tag{11.10}$$

Using identities

$$A^\epsilon = e^{\ln A^\epsilon} = e^{\epsilon \ln A} \simeq 1 + \epsilon \ln A + O(\epsilon^2)$$

$$\Gamma[\epsilon] = \frac{1}{\epsilon} - \gamma_E + O(\epsilon)$$

We will have from above

$$\begin{aligned} \Sigma(p^2) = -e^2 \left(\frac{1}{\epsilon} - \gamma_E + O(\epsilon)\right) \left(1 + \epsilon \ln \frac{4\pi\mu^2}{m^2} + O(\epsilon^2)\right) \\ \times (2-\epsilon) \int_0^1 dx \left(1 - \epsilon \ln \frac{(2+x)m}{(1-x)(1-xp^2/m^2)} + O(\epsilon^2\right) \end{aligned} \tag{11.11}$$

$$= -e^2 \left(\frac{1}{\epsilon} - \gamma_E + \ln \frac{4\pi\mu^2}{m^2}\right) - 2e^2 \int_0^1 dx \ln \frac{(2+x)m^2}{(1-x)(1-xp^2/m^2)} + O(\epsilon) \tag{11.12}$$

The first term on the right is divergence part, in the limit $\epsilon \to 0$, and the second term is a finite part.

11.2 Vacuum polarization

From figure (11.1b), the expression of vacuum polarization is

$$-i\Pi^{\mu\nu}(q^2) = (-ie)^2\mu^{4-d}\int \frac{d^dk}{(2\pi)^d}Tr\left[\gamma^\mu \frac{i(\not{k}+m)}{k^2-m^2}\gamma^\nu \frac{i(\not{q}-\not{k}-m)}{(q-k)^2-m^2}\right] \quad (11.13)$$

$$= e^2\frac{\mu^{4-d}}{(2\pi)^d}\int d^dk \frac{Tr[\gamma^\mu(\not{k}+m)\gamma^\nu(\not{q}-\not{k}-m)]}{(k^2-m^2)((q-k)^2-m^2)} \quad (11.14)$$

$$= e^2\frac{\mu^{4-d}}{(2\pi)^d}\int_0^1 dx\int d^dk \frac{Tr[\gamma^\mu(\not{k}+m)\gamma^\nu(\not{q}-\not{k}-m)]}{[x(k^2-m^2)+(1-x)((q-k)^2-m^2]^2} \quad (11.15)$$

Evaluate the trace

$$Tr[\gamma^\mu(\not{k}+m)\gamma^\nu(\not{q}-\not{k}-m)] = 4[-g^{\mu\nu}(q\cdot k - k^2+m^2)+q^\mu k^\nu + k^\mu(q^\nu - 2k^\nu)$$

The odd-k terms will be eliminated from the numerator of loop momentum integration in (11.15), with even-k denominator. Then we have

$$\begin{aligned}\Pi^{\mu\nu}(q^2) &= ie^2\frac{\mu^{4-d}}{(2\pi)^d}\int_0^1 dx\int d^dk \frac{4[g^{\mu\nu}(k^2-m^2)-2k^\mu k^\nu]}{[x(k^2-m^2)+(1-x)((q-k)^2-m^2)]^2}\\ &= ie^2\frac{\mu^{4-d}}{(2\pi)^d}\int_0^1 dx\int d^dk \frac{4[g^{\mu\nu}(k^2+x^2q^2-m^2)-2k^\mu k^\nu - 2x^2q^\mu q^\nu]}{[k^2-\underbrace{(1-x)(m^2-xq^2)}_{A(x,m^2,q^2)}]^2}\end{aligned} \quad (11.16)$$

After we have shifted the momentum $k \to k+xq$ and done another elimination of the odd-k terms in the numerator. Apply Wick's rotation of the loop momentum, and do the Euclidean momentum integration using spherical coordinate, we will have

$$\begin{aligned}\Pi^{\mu\nu}(q^2) &= -4e^2\int_0^1 dx[g^{\mu\nu}(x^2q^2-m^2)-2x^2q^\mu q^\nu]\frac{\mu^{4-d}}{(2\pi)^d}\frac{\Omega_d}{2}\int d\tilde{k}^2\frac{(\tilde{k}^2)^{d/2-1}}{[\tilde{k}^2+A^2]^2}\\ &\quad +4e^2\int_0^1 dx\frac{\mu^{4-d}}{(2\pi)^d}\frac{\Omega_d}{2}\int d\tilde{k}^2\frac{(\tilde{k}^2)^{d/2-1}(g^{\mu\nu}\tilde{k}^2-2\tilde{k}^\mu\tilde{k}^\nu)}{[\tilde{k}^2+A^2]^2}\end{aligned} \quad (11.17)$$

Using vector integrals

$$\int_0^\infty d\tilde{k}^2(\tilde{k}^2)^{d/2-1}\frac{(\tilde{k}^2)^k}{[\tilde{k}^2+A^2]^n} = \frac{\Gamma[n-d/2-k]\Gamma[d/2+k]}{\Gamma[n]}A^{d/2+k-n}$$

$$\int_0^\infty d\tilde{k}^2(\tilde{k}^2)^{d/2-1}\frac{\tilde{k}^\mu\tilde{k}^\nu}{[\tilde{k}^2+A^2]^n} = \frac{g^{\mu\nu}}{2}\frac{\Gamma[d/2]\Gamma[n-d/2-1]}{\Gamma[n]}A^{d/2+1-n}$$

Then we have from above

$$\begin{aligned}
\Pi^{\mu\nu}(q^2) &= -4e^2 \frac{\mu^{4-d}}{(4\pi)^{d/2}} \frac{\Gamma[2-d/2]}{\Gamma[2]} \\
&\times \int_0^1 dx [g^{\mu\nu}(x^2 q^2 - m^2) - 2x^2 q^\mu q^\nu] A^{d/2-2} \\
&+4e^2 \frac{\mu^{4-d}}{(4\pi)^{d/2}} g^{\mu\nu} \left(\frac{\Gamma[d/2+2]\Gamma[-d/2]}{\Gamma[d/2][\Gamma[n]} \int_0^1 dx A^{d/2} \right. \\
&\left. - \frac{\Gamma[1-d/2]}{\Gamma[2]} \int_)^1 dx x^2 A^{d/2-1} \right) \qquad (11.18)
\end{aligned}$$

IN dimension $d = 4 - 2\epsilon$, we will have

$$\begin{aligned}
\Pi^{\mu\nu}(q^2) = -\frac{e^2}{4\pi^2} \left(\frac{4\pi\mu^2}{m^2} \right)^\epsilon \Gamma[\epsilon] \int_0^1 dx \frac{[g^{\mu\nu}(x^2 q^2 - 2x^2 q^\mu q^\nu]}{[(1-x)(1-xq^2/m^2)]^\epsilon} \\
+\frac{e^2}{4\pi^2} \left(4\pi\mu^2\right)^\epsilon \frac{\Gamma[4-\epsilon]\Gamma[\epsilon-2]}{\Gamma[2-\epsilon]} g^{\mu\nu} \int_0^1 dx [(1-x)(m^2 - xq^2]^{2-\epsilon} \\
-\frac{e^2}{4\pi^2} \Gamma[\epsilon-1] g^{\mu\nu} \int_0^1 dx x^2 [(1-x)(m^2 - xq^2]^{1-\epsilon} \qquad (11.19)
\end{aligned}$$

Using identity

$$\Gamma[\epsilon - 1] = -\frac{\Gamma[\epsilon]}{1-\epsilon}$$

11.3 Vertex correction