12 Renormalization of QED

12.1 QED counter terms

We start with the bare QED Lagrangian

$$\mathcal{L}_0 = \bar{\psi}_0(i\partial\!\!\!/ - m_0)\psi_0 - \frac{1}{4}F_0^{\mu\nu}F_{0\mu\nu} - e_0\bar{\psi}_0\gamma^\mu\psi_0 A_{0\mu} \tag{12.1}$$

Next we define the field renormalizations as

$$\psi_0 = Z_2^{1/2}\psi \text{ and } A_0^\mu = Z_3^{1/2}A^\mu \tag{12.2}$$

and define the charge renoprmalization as

$$e_0 = e\frac{Z_1}{Z_2 Z_3^{1/2}} \tag{12.3}$$

From (2.1), we can write

$$\mathcal{L}_0 = Z_2\bar{\psi}(i\partial\!\!\!/ - m_0)\psi - \frac{Z_3}{4}F^{\mu\nu}F_{\mu\nu} - Z_1 e\bar{\psi}\gamma^\mu\psi A_\mu \tag{12.4}$$

With minimal subtraction scheme, we write

$$Z_1 = 1 + \delta Z_1,\ Z_2 = 1 + \delta Z_2,\ Z_3 = 1 + \delta Z_3 \text{ and } Z_2 m_0 = m + \delta m \tag{12.5}$$

Then we have from above

$$\begin{aligned}
\mathcal{L}_0 &= \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - e\bar{\psi}\gamma^\mu\psi A_\mu \\
&\quad + \bar{\psi}(\delta Z_2 i\partial\!\!\!/ - \delta m)\psi - \frac{\delta Z_3}{4}F^{\mu\nu}F_{\mu\nu} - \delta Z_1 e\bar{\psi}\gamma^\mu\psi A_\mu & (12.6) \\
&= \mathcal{L} + \delta\mathcal{L} & (12.7) \\
\mapsto \mathcal{L} &= \mathcal{L}_0 - \delta\mathcal{L} & (12.8)
\end{aligned}$$

where $\mathcal{L}$ is the renormalized Lagrangian, and $\delta\mathcal{L}$ is the counter term

$$\delta\mathcal{L} = \bar{\psi}(\delta Z_2 i\partial\!\!\!/ - \delta m)\psi - \frac{\delta Z_3}{4}F^{\mu\nu}F_{\mu\nu} - \delta Z_1 e\bar{\psi}\gamma^\mu\psi A_\mu$$

Additional Feynman rules can be assigned for the counter term as

- electron propagator counter term, figure (12.1a), is $i(\delta Z_2 p\!\!\!/ - \delta m)$
- photon propagator counter term, figure (12.1b), is $-ig^{\mu\nu}\delta Z_2 q^2$
- electron-photon vertex counterm, figure (12.1c), is $-ie\delta Z_1\gamma^\mu$

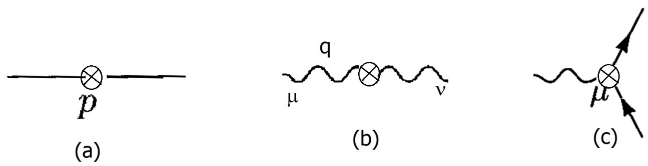

Figure 12.1: QED counter term diagrams.

12.2 Spinor field and mass renormalization

Let us start with the free electron bare propagator

$$\Delta_0(p) = \frac{i}{\not{p} - m_0} = \frac{Z_2 i}{\not{p} - m} = Z_2\Delta(p) \tag{12.9}$$

With interaction, Dyson equation tell us that the interacting bare electron propagator is

$$\Delta_{0I}(p) = \frac{i}{\not{p} - m_0 - \Sigma(\not{p})} \tag{12.10}$$

where $\Sigma(\not{p})$ is electron self energy. Do Taylor expansion in the form

$$\Sigma(\not{p}) = \Sigma(m) + (\not{p} - m)\Sigma'(m) + (\not{p} - m)^2\tilde{\Sigma}(m) \tag{12.11}$$

Let $m = m_0 + \Sigma(m)$, then we have from above

$$\Delta_{0I}(p) = \frac{i}{(\not{p} - m)[1 - \Sigma'(m) - (\not{p} - m)\tilde{\Sigma}(m)]} \xrightarrow{\not{p}=m} \frac{i}{(\not{p} - m)(1 - \Sigma'(m))} \tag{12.12}$$

$$\mapsto Z_2\Delta(p) = \frac{1}{1 - \Sigma'(m)} \simeq 1 + \Sigma'(m) = 1 + \delta Z_2,\ \delta Z_2 = \Sigma'(m) \tag{12.13}$$

$$\delta m = m_0\delta Z_2 = m_0\Sigma'(m) \tag{12.14}$$

12.3 Vector field renormalization

According to bare photon propagator

$$D_0^{\mu\nu}(q) = \frac{-ig^{\mu\nu}}{q^2} = Z_3 D^{\mu\nu}(q) \tag{12.15}$$

With electronic interaction, we have vacuum polarization written in the form

$$\Pi^{\mu\nu}(q^2) = g^{\mu\nu}q^2\bar{\Pi}(q^2) \tag{12.16}$$

From Dyson's equation we will have

$$D_{0I}^{\mu\nu}(q^2) = \frac{-ig^{\mu\nu}}{q^2[1 - \bar{\Pi}(q^2)]} \mapsto Z_3 = \frac{1}{1 - \bar{\Pi}(0)} \simeq 1 + \bar{\Pi}(0) \mapsto \delta Z_3 = \bar{\Pi}(0) \tag{12.17}$$

where we have determined δZ_3 at the pole of photon propagator $q^2 = 0$.

12.4 Charge renormalization

According to the bare vertex

$$\Gamma^0 = -ie_0\gamma^\mu = -ie\frac{Z_1}{Z_2 Z_3^{1/2}}\gamma^\mu = \Gamma^\mu \tag{12.18}$$

We can simplify this be setting $Z_1 = Z_2$, then we have

$$e^2 = Z_3 e_0^2 = \frac{e_0^2}{1-\bar{\Pi}(0)} \tag{12.19}$$

12.5 Ward identity

Let us determine three-point function on momentum space, with the conserved current $J^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x) \to \partial_\mu J^\mu = 0$,

$$C^\mu(p,k) = \int d^4x \int d^4y e^{-ip\cdot x - ik\cdot y}\langle 0|T[J^\mu(x)\psi(y)\bar{\psi}(0)]|0\rangle \tag{12.20}$$

We observe that

$$p_\mu C^\mu(p,k) = \int d^4x \int d^4y i\partial_{x\mu}\left(e^{-ip\cdot x - ik\cdot y}\right)\langle 0|T[J^\mu(x)\psi(y)\bar{\psi}(0)]|0\rangle \tag{12.21}$$

$$\xrightarrow{\textit{int. by part}} -i\int d^4x \int d^4y e^{-ip\cdot x - ik\cdot y}\partial_{x\mu}\langle 0|T[J^\mu(x)\psi(y)\bar{\psi}(0)]|0\rangle \tag{12.22}$$

Since

$$\begin{aligned} T[J^\mu(x)\psi(y)\bar{\psi}(0)] &= \theta(x^0-y^0)J^\mu(x)\psi(y)\bar{\psi}(0) + \theta(y^0-x^0)\psi(y)J^\mu(x)\bar{\psi}(0) \\ &+\theta(y^0-0)J^\mu(x)\psi(y)\bar{\psi}(0) - \theta(0-y^0)J^\mu(x)\bar{\psi}(0)\psi(y) \\ &+\theta(x^0-0)\psi(y)J^\mu(x)\bar{\psi}(0) - \theta(0-x^0)\bar{\psi}(0)J^\mu(x)\psi(y) \end{aligned}$$

Then we have

$$\begin{aligned} \partial_{x\mu}T[A^\mu(x)\psi(y)\bar{\psi}(0)] &= T[(\partial_{x\mu}J^\mu(x))\psi(y)\bar{\psi}(0)] + T[\delta(x^0-y^0)\{J^0(x),\psi(y)\}\bar{\psi}(0)] \\ &-T[\delta(x^0)\{J^0(x),\bar{\psi}(0)\}\psi(y)] \end{aligned}$$

Insertion into (12.22), we have

$$\begin{aligned} p_\mu C^\mu(p,k) &= -i\int d^4x \int d^4y e^{-ip\cdot x - ik\cdot y}\langle 0|\delta(x^0-y^0)\{J^0(x),\psi(y)\}\bar{\psi}(0)|0\rangle \\ &+i\int d^4x \int d^4y e^{-ip\cdot x - ik\cdot y}\langle 0|\delta(x^0)\{J^0(x),\bar{\psi}(0)\}\psi(y)|0\rangle \end{aligned} \tag{12.23}$$

Since

$$\pi = i\bar{\psi}\gamma^0 = i\psi^\dagger \mapsto \{\psi(x),\pi(y)\}_{x^0=y^0} = i\{\psi(x),\psi^\dagger(y)\}_{x^0=y^0} = i\delta^{(3)}(\vec{x}-\vec{y})$$

$$J^0 = \bar{\psi}\gamma^0\psi = \psi^\dagger\psi \mapsto \{J^0(x),\psi(y)\}_{x^0=y^0} = \{\psi^\dagger(x)\psi(x),\psi(y)\}_{x^0=y^0}$$

$$= \{\psi^\dagger(x), \psi(y)\}_{x^0=y^0}\psi(x) = \delta^{(3)}(x-y)\psi(x)$$

$$\{J^0(x), \bar{\psi}(0)]\}_{x^0=0} = \{\psi^\dagger(x)\psi(x), \psi^\dagger(0)\gamma^0]_{x^0=0} = \delta^{(3)}(x)\bar{\psi}(x)$$

Insertion into (12.23), we get

$$\begin{aligned} p_\mu C^\mu(p,k) = -i\int d^4x e^{-i(p+k)\cdot x}\langle 0|T[\psi(x)\bar{\psi}(0)]|0\rangle \\ +i\int d^4x e^{-ik\cdot x}\langle 0|T[\bar{\psi}(0)\psi(x)]|0\rangle \end{aligned} \tag{12.24}$$

$$\mapsto ip_\mu C^\mu(p,k) = \Delta(p+k) - \Delta(k), \text{ with } \Delta(p) = \frac{i}{\not{p}-m} \tag{12.25}$$

This is called *Ward identity*, which tell us how the three point function is related to the two point function in term of the conserve energy-momentum current.

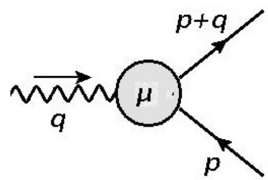

Figure 12.2: Ward identity of QED vertex.

From figure (12.2), we define the vertex function as

$$\begin{aligned} -iq_\mu\Gamma^\mu(p.q) &= \Delta^{-1}(p+q)(-iq_\mu C^\mu(p,q))\Delta^{-1}(q) \\ &= \Delta^{-1}(p+q) - \Delta^{-1}(q) \end{aligned} \tag{12.26}$$

From the vertex renormalization above, we have

$$eZ_2\Gamma^\mu(q\to 0) = e\gamma^\mu, \; \Delta(p) = \frac{iZ_2}{\not{p}-m} \tag{12.27}$$

$$-iq_\mu Z_1^{-1}\gamma^\mu = -iZ_2^{-1}(\not{p}+\not{q}-m) - (-i)Z_2^{-1}(\not{p}-m) \tag{12.28}$$

$$\mapsto Z_1 = Z_2 \tag{12.29}$$

So that our assumption of this equality comes from Ward identity.

12.6 QED beta function