1 Eliashberg Theory

Electron-phonon Hamiltonian, without Coulomb interaction, is

$$\begin{aligned} H &= H_e + H_p + H_{ep} \\ &= \sum_{k,\sigma} c^\dagger_{k,\sigma} c_{k,\sigma} + \sum_q \omega_q b^\dagger_q b_q + \sum_{k,q,\sigma} g_q (b_q + b^\dagger_{-q}) c^\dagger_{k+q,\sigma} c_{k,\sigma} \end{aligned} \tag{1}$$

where $\xi_k = \epsilon_k - \mu$.

1.1 Nambu Green's function

Nambu spinor

$$\Psi_k = \begin{pmatrix} {}_{k\uparrow} \\ c^\dagger_{-k\downarrow} \end{pmatrix}, \ \Psi^\dagger_k = \begin{pmatrix} c^\dagger_{k\uparrow} & c_{-k\downarrow} \end{pmatrix} \tag{2}$$

Rewrite electron Hamiltonian

$$H_e = \sum_k \xi_k \Psi^\dagger_k \tau^3 \Psi_k, \ H_{ep} = \sum_{k,q} g_q (b_q + b^\dagger_{-q}) \Psi^\dagger_{k+q} \tau^3 \Psi_k \tag{3}$$

Thermal Green's function is defined in the form

$$\begin{aligned} \tilde{G}(k,\tau) &= -\langle T_\tau [\Psi_k(\tau) \Psi^\dagger_k(0)] \rangle \\ &= -\begin{pmatrix} \langle T_\tau [c_{k\uparrow}(\tau) c^\dagger_{k\uparrow}(0)] \rangle & \langle T_\tau [c_{k\uparrow}(\tau) c_{-k\downarrow}(0)] \rangle \\ \langle T_\tau [c^\dagger_{-k\downarrow}(\tau) c^\dagger_{k\uparrow}(0)] \rangle & \langle T_\tau [c^\dagger_{-k\downarrow}(\tau) c_{-k\downarrow}(0)] \rangle \end{pmatrix} \\ &= \begin{pmatrix} G(k,\tau) & F(k,\tau) \\ -F(k,-\tau) & -G(k,-\tau) \end{pmatrix} \end{aligned} \tag{4}$$

where $G(k,\tau)$ is the normal Green's function, while $F(k,\tau)$ is anomalous Green's function, defined by Gorkov. Their time-Fourier transformations are

$$\left\{ \begin{array}{c} G(k,i\nu_n) \\ F(k,i\nu_n) \end{array} \right\} = \frac{1}{\beta} \sum_n e^{-i\nu_n \tau} \left\{ \begin{array}{c} G(k,i\nu_n) \\ F(k,i\nu_n) \end{array} \right\} \tag{5}$$

$$\left\{ \begin{array}{c} G(k,i\nu_n) \\ F(k,i\nu_n) \end{array} \right\} = \int_0^\beta d\tau e^{i\nu_n \tau} \left\{ \begin{array}{c} G(k,i\nu_n) \\ F(k,i\nu_n) \end{array} \right\} \tag{6}$$

We will have form (4)

$$\tilde{G}(k, i\nu_n) = \begin{pmatrix} G(k, i\nu_n) & F(k, i\nu_n) \\ F(k, -i\nu_n) & G(k, -i\nu_n) \end{pmatrix} \tag{7}$$

Free electron Green's function

$$\tilde{G}_0(k, i\nu_n) = \begin{pmatrix} G_0(k, i\nu_n) & 0 \\ 0 & G_0(k, -i\nu_n) \end{pmatrix}, \; G_0(k, i\nu_n) = \frac{1}{i\nu_n - \xi_k} \tag{8}$$

Interacting Green's function is determined form Dyson-Gorkov equation

$$\tilde{G}^{-1}(k, i\nu_n) = \tilde{G}_0^{-1}(k, i\nu_n) - \tilde{\Sigma}(k, i\nu_n) \tag{9}$$

where $\tilde{\Sigma}(k, i\nu_n)$ is the electron self-energy, from phonon interaction. Its expression is

$$\tilde{\Sigma}(k, i\nu_n) = \frac{1}{\beta} \sum_{n'} \int \frac{d^3k'}{(2\pi)^3} g^2_{k-k'} D(k - k', i\nu_n - i\nu_{n'}) \tau^3 \tilde{G}(k', i\nu_{n'}) \tau^3 \tag{10}$$

Its generic form should be

$$\tilde{\Sigma}(k, i\nu_n) = \begin{pmatrix} \Sigma(k, i\nu_n) & \Phi(k, i\nu_n) \\ \Phi^*(k, i\nu_n) & \Sigma^*(k, i\nu_n) \end{pmatrix} \tag{11}$$

where $\Sigma(K, i\nu_n)$ is normal self-energy, and $\Phi(k, i\nu_n)$ is anomalous self-energy. In normal system, we extract the imaginary part and real part of the self-energy in the form

$$\Sigma(k, i\nu_n) - \Sigma^*(k, i\nu_n) = 2i\nu_n(1 - Z(k, i\nu_n)) \tag{12}$$

$$\Sigma(k, i\nu_n) + \Sigma^*(k, i\nu_n) = 2\chi(k, i\nu_n) \tag{13}$$

and rewrite (10) in the form

$$\begin{aligned} \tilde{\Sigma}(k, i\nu_n) &= i\nu_n(1 - Z(k, i\nu_n)\tau^3 + \chi(k, i\nu_n)\tau^1 \\ &\quad + \phi_1(k, i\nu_n)\tau^1 + \phi_2(k, i\nu_n)\tau^2 \qquad (14) \\ &= \begin{pmatrix} i\nu_n(1 - Z) + \chi & \Phi \\ \Phi^* & -i\nu_n(1 - Z) + \chi \end{pmatrix} \qquad (15) \end{aligned}$$

We will observe that $\Phi = \phi_1 - i\phi_2$ and $\Phi^* = -\Phi$, a pure imaginary.
Since

$$\tilde{G}_0^{-1}(k, i\nu) = \begin{pmatrix} i\nu_n - \xi_k & 0 \\ 0 & -i\nu_n - \xi_k \end{pmatrix} \tag{16}$$

Back insertion (15) and (16) into (9), we get

$$\tilde{G}^{-1}(k, i\nu_n) = \begin{pmatrix} i\nu_n Z - \xi_k - \chi & -\Phi \\ -\Phi^* & -i\nu_n Z - \xi_k - \chi \end{pmatrix} \tag{17}$$

Its inversion is

$$\tilde{G}(k, i\nu_n) = \frac{1}{\Omega(k, i\nu_n)} \begin{pmatrix} -i\nu_n Z - (\xi_k + \chi) & \Phi^* \\ \Phi & i\nu_n Z - (\xi_k + \chi) \end{pmatrix} \tag{18}$$

where

$$\Omega(k, i\nu_n) = \det \tilde{G} = (\xi_k + \chi)^2 - (i\nu_n Z)^2 - |\Phi|^2 \tag{19}$$

with a known solution of $\xi_k = \epsilon_k - \mu$.

1.2 Eliashberg equations

The electronic spectrum is determined from the poles of $\tilde{G}(k, i\nu_n)$, that is from a condition $\Omega(k, i\nu_n) = 0$. To know this we have to have solutions of Z, χ and Φ. They can be determined from (10) with self-consistent insertion of $\tilde{G}(k, i\nu_n)$. After some algebra, we will have

$$\Phi(k, i\nu_n) = -\frac{1}{\beta} \sum_{n'} \int \frac{d^3k'}{(2\pi)^3} g^2_{k-k'} D(k - k', i\nu_n - i\nu_{n'}) \frac{\Phi(k', i\nu_{n'})}{\Omega(k', i\nu_{n'})} \tag{20}$$

$$Z(k, i\nu_n) = 1 - \frac{1}{\beta} \sum_{n'} \int \frac{d^3k'}{(2\pi)^3} g^2_{k-k'} D(k - k', i\nu_n - i\nu_{n'}) \frac{i\nu_{n'}}{i\nu_n} \frac{Z(k, i\nu_{n'})}{\Omega(k', i\nu_{n'})} \tag{21}$$

$$\chi(k, i\nu_n) = -\frac{1}{\beta} \sum_{n'} \int \frac{d^3k'}{(2\pi)^3} g^2_{k-k'} D(k - k', i\nu_n - i\nu_{n'}) \frac{\xi_{k'} + \chi(k', i\nu_{n'})}{\Omega(k', i\nu_{n'})} \tag{22}$$

Note that $\Phi = 0$ is always trivial solution of (20), the other solutions are self-consistent.